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Local E_{11}

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Abstract: We give a method of deriving the field-strengths of all massless and massive maximal supergravity theories in any dimension starting from the Kac-Moody algebra E_{11}. Considering the subalgebra of E_{11} that acts on the fields in the non-linear realisation as a global symmetry, we show how this is promoted to a gauge symmetry enlarging the algebra by the inclusion of additional generators. We show how this works in eleven dimensions, and we call the resulting enlarged algebra E_{11}^{local}. Torus reduction to D dimensions corresponds to taking a subalgebra of E_{11}^{local}, called $E_{11,D}^{\text{local}}$, that encodes the full gauge algebra of the corresponding D-dimensional massless supergravity. We show that each massive maximal supergravity in D dimensions is a non-linear realisation of an algebra $\tilde{E}_{11,D}^{\text{local}}$. We show how this works in detail for the case of Scherk-Schwarz reduction of IIB to nine dimensions, and in particular we show how $\tilde{E}_{11,9}^{\text{local}}$ arises as a subalgebra of the algebra $E_{11,10B}^{\text{local}}$ associated to the ten-dimensional IIB theory. This subalgebra corresponds to taking a combination of generators which is different to the massless case. We then show that $\tilde{E}_{11,D}^{\text{local}}$ appears as a deformation of the massless algebra $E_{11,D}^{\text{local}}$ in which the commutation relations between the E_{11} and the additional generators are modified. We explicitly illustrate how the deformed algebra is constructed in the case of massive IIA and of gauged five-dimensional supergravity. These results prove the naturalness and power of the method.

Keywords: Extended Supersymmetry, Gauge Symmetry, Space-Time Symmetries, Supergravity Models

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1 Introduction

It has been conjectured in [1] that eleven dimensional supergravity could be extended so as to have a non-linearly realised infinite-dimensional Kac-Moody symmetry called E_{11}, whose Dynkin diagram is shown in figure 1. In a non-linear realisation the algebra used to construct it is realised as a rigid symmetry. However, in the eleven dimensional supergravity theory all the symmetries are local. In this paper we will propose a non-linear realisation in which E_{11} symmetries become local. To put this work in context it will be useful to list some of the main developments of the E_{11} programme which are relevant for this paper.

Eleven dimensional supergravity itself can be formulated as a non-linear realisation based on an algebra that includes generators with non-trivial Lorentz character [2]. To find the precise dynamics one takes the simultaneous non-linear realisation of this algebra with the conformal group. This naturally gives rise to both a 3-form and a 6-form fields and the resulting field equations are first order duality relations, whose divergence reproduces the 3-form second-order field equations of 11-dimensional supergravity provided on chooses one constant. The eleven-dimensional gravity field describes non-linearly $GL(11,\mathbb{R})$, which is a subalgebra of this algebra. Indeed, gravity in D dimensions can be described as a non-linear realisation of the closure of the group $GL(D,\mathbb{R})$ with the conformal group [2], as was originally shown in the four dimensional case in [3].

E_{11} first arose as the smallest Kac-Moody algebra which contains the algebra found in the non-linear realisation above. This E_{11} algebra is infinite-dimensional, and the E_{11} non-linear realisation contains an infinite number of fields with increasing number of indices. The first few fields are the graviton, a three form, a six form and a field which has the right spacetime indices to be interpreted as a dual graviton. This is the field content of eleven dimensional supergravity, and keeping only the first three of these fields one finds that the non-linear realisation of E_{11} reduces to the construction discussed in the first point and so results in the dynamics of this theory [1]. Theories in D dimensions arise from the E_{11} non-linear realisation by choosing a suitable $GL(D,\mathbb{R})$ subalgebra, which is associated with D-dimensional gravity. The A_{D-1} Dynkin diagram of this subalgebra, called the gravity line, must include the node labelled 1 in the Dynkin diagram of figure 1. In ten dimensions there are two possible ways of constructing this subalgebra, and the corresponding non-linear realisations give rise to two theories that contain the fields of the IIA and IIB supergravity theories and their electromagnetic duals [1, 4]. Below ten dimensions, there is a unique choice for this subalgebra, and this corresponds to the fact that massless maximal supergravity theories in dimensions below ten are unique. Again, the non-linear realisation in each case describes, among an infinite set of other fields, the fields of the corresponding supergravity and their electromagnetic duals. In each dimension, the part of the E_{11} Dynkin diagram which is not connected to the gravity line corresponds to the internal hidden symmetry of the D dimensional theory. This not only reproduces all the hidden symmetries found long ago in the dimensionally reduced theories, but it also gives an eleven-dimensional origin to these symmetries.

All the maximal supergravity theories mentioned so far are massless in the sense that no other dimensional parameter other than the Planck scale is present. In fact, even this parameter can be absorbed into the fields such that it is absent from the equations of motion. There are however other theories that are also maximal, i.e. invariant under 32 supersymmetries, but are massive in the sense that they possess additional dimensionful parameters. These can be viewed as deformations of the massless maximal theories. However, unlike the massless maximal supergravity theories they can not in general be obtained by a process of dimensional reduction and in each dimension they have been determined by analysing the deformations that the corresponding massless maximal supergravity admits. With the exception of the one deformation allowed for type IIA supergravity in ten dimensions, called Roman's theory, all the massive maximal supergravities possess a local gauge symmetry carried by vector fields that is a subgroup of the symmetry group G of the corresponding maximal supergravity theory, and are therefore called gauged supergravities. In general these theories also have potentials for the scalars fields which contain the dimensionful parameters as well as a cosmological constant. In recent years there have been a number of systematic searches for gauged maximal supergravity theories and in particular in nine dimensions and in dimension from seven to three all such theories have been classified [5–7].

It will be useful to recall how E_{11} has from a very different perspective lead to the classification of gauged supergravities that agrees with these results and how the E_{11} formulation of the gauged supergravity theories has lead to new work in these theories. The cosmological constant of ten-dimensional Romans IIA theory [8] can be described as the dual of a 10-form

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field-strength [9], and the supersymmetry algebra closes on the corresponding 9-form potential [10]. The Romans theory was found to be a non-linear realisation [11] which includes all form fields up to and including a 9-form with a corresponding set of generators. This 9-form is automatically encoded in the non-linear realisation of E_{11} [12]. From the eleven dimensional E_{11} theory it arises as the dimensional reduction of the eleven-dimensional field $A_{a_1\ldots a_{10},(bc)}$ in the irreducible representation of $GL(11,\mathbb{R})$ with ten antisymmetric indices $a_1\ldots a_{10}$ and two symmetric indices b and c. Therefore E_{11} not only contains Romans IIA, but it also provides it for the first time with an eleven-dimensional origin [13].

By studying the eleven-dimensional fields of the E_{11} non-linear realisation, one can determine all the forms, i.e. fields with completely antisymmetric indices, that arise from dimensional reduction to any dimension [14]. In particular, in addition to all the lower rank forms, this analysis gives all the $D-1$-forms and the D-forms in D dimensions. The list of all form fields obtained in this way for all supergravity theories is given in table 1. The $D-1$ and D-forms predicted by E_{11} can also be derived in each dimension separately [15]. The $D-1$-forms have D-form field strengths, that are related by duality to the mass deformations of gauged maximal supergravities, and the E_{11} analysis shows perfect agreement with the complete classification of gauged supergravities performed in [6, 7]. Therefore E_{11} not only contains all the possible massive deformations of maximal supergravities in a unified framework, but it also provides an eleven-dimensional origin to all of them. Indeed, while some gauged supergravities were known to be obtainable using dimensional reduction of ten or eleven dimensional supergravities, this was not generically the case. As a result the gauged supergravities were outside the framework of M-theory as it is usually understood.

One striking feature of the E_{11} formulation of massless or massive supergravity theories is that it includes fields together with all their dual fields. The presence of the dual forms is essential to formulate the field equations as duality relations. Some dual forms have been introduced in the past in an ad-hoc way beginning with [16], but it is only with E_{11} that they have arisen from an underlying principle. Indeed, the forms of table 1 were proposed in [14, 15] to play a crucial role in gauged supergravities, the $D-1$ forms classifying the gauged supergravities and the lower forms providing a chain of form fields that occur in the duality relations. This is compatible with the structure of the gauge algebra arising in gauged supergravities, in which one is forced to introduce a $p+1$ form to close the gauge algebra of a p form, thus determining a hierarchy of forms [17]. For the cases in which this latter method has been subsequently used to compute the hierarchy of forms, the results are precisely in agreement with E_{11} [18], and indeed the presence of the forms given in table 1 has now been systematically adopted by those studying gauged supergravities.

All in all there is considerable evidence for an E_{11} symmetry in the low energy limit of what is often called M theory. The above evidence concerns the adjoint representation of E_{11}, or the part of the non-linear realisation that involves the fields associated with the E_{11} generators. However, there is also the question of how space-time is encoded in the theory. In the non-linear realisations mentioned above the generator of space-time translations P_a was introduced by hand in order to encode the coordinates of space-time. From the beginning it was understood that this was an ad-hoc step that did not respect the E_{11} symmetry. It was subsequently proposed [19] that one could include an E_{11} multiplet

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D	G	1-forms	2-forms	3-forms	4-forms	5-forms	6-forms	7-forms	8-forms	9-forms	10-forms
10A	$\mathbb{R}^+$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$		$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$ $\mathbf{1}$
10B	$\mathrm{SL}(2,\mathbb{R})$		$\mathbf{2}$		$\mathbf{1}$		$\mathbf{2}$		$\mathbf{3}$		$\mathbf{4}$ $\mathbf{2}$
9	$\mathrm{SL}(2,\mathbb{R})\times\mathbb{R}^+$	$\mathbf{2}$ $\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$ $\mathbf{1}$	$\mathbf{3}$ $\mathbf{1}$	$\mathbf{3}$ $\mathbf{2}$	$\mathbf{4}$ $\mathbf{2}$ $\mathbf{2}$	
8	$\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$	$(\overline{\mathbf{3}},\mathbf{2})$	$(\mathbf{3},\mathbf{1})$	$(\mathbf{1},\mathbf{2})$	$(\overline{\mathbf{3}},\mathbf{1})$	$(\mathbf{3},\mathbf{2})$	$(\mathbf{8},\mathbf{1})$ $(\mathbf{1},\mathbf{3})$	$(\mathbf{6},\mathbf{2})$ $(\overline{\mathbf{3}},\mathbf{2})$	$(\mathbf{15},\mathbf{1})$ $(\mathbf{3},\mathbf{3})$ $(\mathbf{3},\mathbf{1})$ $(\mathbf{3},\mathbf{1})$		
7	$\mathrm{SL}(5,\mathbb{R})$	$\overline{\mathbf{10}}$	$\mathbf{5}$	$\overline{\mathbf{5}}$	$\mathbf{10}$	$\mathbf{24}$	$\overline{\mathbf{40}}$ $\overline{\mathbf{15}}$	$\mathbf{70}$ $\mathbf{45}$ $\mathbf{5}$			
6	$\mathrm{SO}(5,5)$	$\mathbf{16}$	$\mathbf{10}$	$\overline{\mathbf{16}}$	$\mathbf{45}$	$\mathbf{144}$	$\mathbf{320}$ $\overline{\mathbf{126}}$ $\mathbf{10}$				
5	$E_{6(+6)}$	$\mathbf{27}$	$\overline{\mathbf{27}}$	$\mathbf{78}$	$\mathbf{351}$	$\overline{\mathbf{1728}}$ $\overline{\mathbf{27}}$					
4	$E_{7(+7)}$	$\mathbf{56}$	$\mathbf{133}$	$\mathbf{912}$	$\mathbf{8645}$ $\mathbf{133}$						
3	$E_{8(+8)}$	$\mathbf{248}$	$\mathbf{3875}$ $\mathbf{1}$	$\mathbf{147250}$ $\mathbf{3875}$ $\mathbf{248}$							

Table 1. Table giving the representations of the symmetry group G of all the forms of maximal supergravities in any dimension [14]. The 3-forms in three dimensions were determined in [15].

of generators which had as its lowest component the generator of space-time translations. This is just the fundamental representation of E_{11} associated with the node labelled 1 in the Dynkin diagram of figure 1 and it is denoted by l. A method of constructing the gauged supergravities was given in reference [20] using E_{11} and the l multiplet of generators. Indeed as an example all the gauged supergravity generators in five dimensions were derived from this viewpoint. This reference also contains a review of the evidence for the l multiplet as the multiplet of brane charges and a table of its low level content in dimensions three and above. In this context there has been a recent interesting paper [21] which keeps the scalar charges in the l multiplet for the seven-dimensional maximal supergravity and still finds diffeomorphism invariance in seven dimensions.

What was not clear from this method was how the global E_{11} symmetries would become local and this is the subject of this paper. In the context of purely gravity this was achieved long ago in reference [3] by taking the simultaneous non-linear realisation of $IGL(4,\mathbb{R})$ with the conformal group in four dimensions. As mentioned above, if one took the non-linear realisation of E_{11} at low levels, that is to include the six form generator, took only the Lorentz group as the local subgroup and the simultaneous non-linear realisation with the conformal group, then the dynamics predicted by the non-linear realisation is just the maximal supergravity theory in eleven dimensions. This can be seen by realising that the low level E_{11} algebra [1] is just that used in reference [2] to construct the eleven dimensional supergravity theory as a non-linear realisation once one includes the conformal group. The effect of latter is that it makes not only the space-time translations a local symmetry but also turns the shifts associated with form fields into gauge transformations [2]. However, it is not clear how to combine the conformal group with the algebra formed from E_{11} and the l multiplet. In particular how to extend the action of the conformal group on the usual coordinates of space-time to include the other coordinates encoded in the l multiplet.

In this paper we will not use the conformal group, but rather add the generators that the closure of this algebra with E_{11} would generate. We will also not use the generators from the l multiplet, but only the space-time translations P_a. The prototype example of this mechanism was given long ago for the case of Yang-Mills theory [22]. Essentially one takes an algebra that contains the generators P_a and the Yang Mills generators Q^α, as well as the generators $R^{a,\alpha}$ for which the gauge fields are Goldstone bosons and an infinite number of generators $K^{a_1\ldots a_n,\alpha}$, symmetric in their spacetime indices, which do not commute with P_a and whose role is to make the rigid symmetry generated by $R^{a,\alpha}$ local. We will review this construction later on in this introduction.

We will first show the analogous mechanism for pure gravity. In particular, we will show how to construct Einstein's theory of gravity using a non-linear realisation which takes as its underlying algebra one that consists of $IGL(D,\mathbb{R})$ and an infinite set of additional generators whose effect will to promote the rigid $IGL(D,\mathbb{R})$ to be local. The generators P_a lead in the non-linear realisation to the coordinates of space-time while the Goldstone boson for $GL(D,\mathbb{R})$ is the vierbein which is subject to local Lorentz transformations. The infinite number of additional generators lead to local translations, that is general coordinate transformations, but to no new fields in the final theory as their Goldstone fields are solved in terms of the graviton field using a set of invariant constraints placed on the Cartan forms. This is an example of what has been called the inverse Higgs effect [23]. The unique theory resulting from this non-linear realisation with only two space-time derivatives is Einstein's theory up to a possible cosmological term. In this case one can see that the additional generators we have added are just those found by taking the closure of $IGL(D,\mathbb{R})$ with the conformal group.

We will then generalise this procedure to E_{11} at low levels. We take the algebra, called $E_{11}^{\rm local}$ consisting of non-negative level E_{11} generators, the generators P_a and an infinite number of additional generators. While the latter lead in the final result to no new Goldstone fields they do result in all the low level E_{11} symmetries becoming local, thus we find general coordinate transformations and gauge transformations for all the form fields. For the eleven dimensional theory, space-time arises in the group element due to the P_a generators, however, for lower dimensional theories we will take space-time to be not only the translation operator P_a for that dimension but also certain other Lorentz scalar charges that include the translation operators for the dimensionally reduced generators, in effect we take only the Lorentz scalar part of the l multiplet. As we add just the spacetime translations rather than the whole l multiplet we will take P_a to commute with the non-negative level generators of E_{11}. The price for proceeding in this way is that we are working with only the non-negative level generators of E_{11} and we have essentially thrown out the negative level generators. We show that the non-linear realisation of the algebra $E_{11}^{\rm local}$ describes at low levels in eleven dimensions the 3-form and the 6-form of the eleven dimensional supergravity theory with all their gauge symmetries. This can be thought of as equivalent to taking the non-linear realisation of E_{11} at low levels and taking the simultaneous non-linear realisation with the conformal group as was discussed earlier [1, 2], but here the procedure is more transparent.

We then consider the formulation of lower dimensional maximal gauged supergravity theories from the viewpoint of the enlarged algebra $E_{11,D}^{\rm local}$. The D refers to the fact that although we take the same non-negative level E_{11} generators and generators P_a, the infinite number of additional generators we take vary from dimension to dimension. We first consider as a toy model the Scherk-Schwarz dimensional reduction of the IIB supergravity theory from this viewpoint. We begin with an algebra consisting of $E_{11,10B}^{\rm local}$ and take the ten dimensional space-time to arise from an operator $\tilde{Q}$ which is constructed from $Q = P_9$ and part of the $\mathrm{SL}(2,\mathbb{R})$ symmetry of the theory. This means that the 10th direction of space-time is twisted to contain a part in the $\mathrm{SL}(2,\mathbb{R})$ coset symmetry of the theory. This non-linear realisation gives a nine dimensional gauged supergravity. We observe that not all of the algebra $E_{11,10B}^{\rm local}$ is essential for the construction of the gauged supergravity in nine dimensions, but only an algebra which we call $\tilde{E}_{11,9}^{\rm local}$ which is the subalgebra of $E_{11,10B}^{\rm local}$ that commutes with $\tilde{Q}$. Its generators are non-trivial combinations of E_{11} generators and the additional generators and in general the generators of $\tilde{E}_{11,9}^{\rm local}$ have non-trivial commutation relations with nine dimensional space-time translations. Although the subalgebra $\tilde{E}_{11,9}^{\rm local}$ appears to be a deformation of the original E_{11} algebra and the space-time translations we have not changed the original commutators, but rather the new algebra arises due to the presence of the additional generators which are added to the E_{11} generators.

However, we then show that one can find the algebra $\tilde{E}_{11,9}^{\rm local}$ without carrying out all the above steps. Given the non-trivial relation between the lowest non-trivial positive level generator of $\tilde{E}_{11,9}^{\rm local}$ and the nine dimensional space-time translations one can derive the rest of the algebra $\tilde{E}_{11,9}^{\rm local}$ simply using Jacobi identities. This algebra determines uniquely all the field strengths of the theory, and thus one finds a very quick way of deriving the gauged supergravity theory.

This picture applies to all gauged supergravity theories, as one can easily find the algebra $\tilde{E}_{11,D}^{\rm local}$ without using its derivation from $E_{11}^{\rm local}$ and this provides a very efficient method of constructing all gauged supergravities. We illustrate how this works by constructing the massive IIA theory as well as all the gauged maximal supergravities in five dimensions.

Finally, we consider how this construction generalises to the fields with mixed symmetry, i.e. not completely antisymmetric, of E_{11} and in general of any non-linear realisation of a very-extended Kac-Moody algebra. We will consider as a prototype of such fields the dual graviton in four dimensions, which is a field A_{ab} symmetric in its two spacetime indices. We will show that if one tries to promote the global shift symmetry of the dual graviton field to a gauge symmetry, one finds that this is not compatible with the E_{11} algebra. The solution of this problem is that actually E_{11} forces to include additional generators, whose role is to enlarge the gauge symmetry of the dual graviton so that one can gauge away the field completely. We show this first for the simpler case of the non-linear realisation of the Kac-Moody algebra A_1^{+++} in four dimensions. We then consider the case of E_{11} in four dimensions. For simplicity in this case we neglect the gravity generators, and we still find that even considering consistency conditions involving only the generators associated to the form fields and those associated to the dual graviton, one is forced to include additional generators for the dual graviton that generate a local symmetry that gauges away the dual graviton completely. We claim that this picture generalises to all mixed symmetry fields in any dimension. It is important to stress that the dynamics is compatible with this result. Indeed, while the field strengths of the antisymmetric fields are first order in derivatives, and therefore one needs fields and dual fields to construct duality relations which are first order equations for these fields, the gravity Riemann tensor is at second order in derivatives and thus there is no need of a dual field to construct its equation of motion.

It will be helpful to recall some facts about non-linear realisations. A non-linear realisation of a group G with respect to a subgroup H is by definition a theory invariant under the two separate transformations

$$g(x) \to g_0 g(x), \quad g(x) \to g(x)h(x) \tag{1.1}$$

where $g \in G$, $g_0 \in G$ while $h \in H$. The dependence on the generic symbol x signifies which group elements dependence on the coordinates of the space-time. For the case of an internal symmetry the space-time dependence is incorporated by hand. However, in this paper space-time will arise naturally in that its associated generators are part of the Lie algebra of the group G. Indeed, a part of the group element is just space-time viewed as a coset. We note that the h transformations depend on the space-time coordinates so can be said to be local, unlike the rigid g_0 transformations. Working with the most general group element g we must then find a theory that is invariant under both g_0 and local h transformations.

It is often more transparent to use the h transformations to choose $g(x)$ to be of a particular form, that is choose coset representatives. If one does this then when making a rigid g_0 transformation one finds a group element $g_0 g$ which is in general not one of the chosen coset representatives. To rectify this one must make a compensating h_c that depends on g_0 and the original coset representative $g(x)$. That is $g \to g' = g_0 g h_c^{-1}$ where both g and g' are chosen coset representatives.

The problem of finding the invariant dynamics is most often solved by using the Cartan forms $\mathcal{V} = g^{-1}dg$. This is obviously invariant under rigid g_0 transformations and transforms as

$$\mathcal{V} \to h^{-1}\mathcal{V}h + h^{-1}dh \tag{1.2}$$

under local h transformations. We note that $g^{-1}dg = dx \cdot g^{-1}\partial g$ is invariant but $g^{-1}\partial g$ is not as the coordinates of space-time x transform under g_0 transformations. To be more explicit we consider a group that contains the generators L_N and we denote the remaining generators by the generic symbol T^*. We will assume that the generators L_N from a representation of the T^*'s. The general group element is of the form

$$g = e^{x \cdot L} e^{\phi(x) \cdot T}. \tag{1.3}$$

We recognise x as the coordinates and ϕ as the fields. The local subgroup can be used to set some of the fields ϕ to zero. The discussion below holds if one makes this choice or

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work with the general group element. The Cartan forms can be written as

$$\mathcal{V} = g^{-1}dg = dx^{\Pi}E_{\Pi}{}^{N}L_N + dx^{\Pi}G_{\Pi,*}T^{*}. \tag{1.4}$$

Since $\mathcal{V}$ is invariant under $g \to g_0 g$ it follows that each of the coefficients of the above generators is invariant, that is $dx^{\Pi}E_{\Pi}{}^{N}$ and $dx^{\Pi}G_{\Pi,*}$ are invariant. However, dx^{Π} does transform under g_0 and so $E_{\Pi}{}^{N}$ and $G_{\Pi,*}$ are not invariant. To find quantities that only transform under the local subalgebra we can rewrite $\mathcal{V}$ as

$$\mathcal{V} = g^{-1}dg = dx^{\Pi}E_{\Pi}{}^{N}(L_N + G_{N,*}T^{*}), \tag{1.5}$$

where we recognise that $G_{N,*} = (E^{-1})_N{}^{\Pi}G_{\Pi,*}$. It follows that $G_{N,*}$ are inert under g_0 transformations and just transform under local transformations. As such they are useful quantities with which to construct the dynamics as one must now only solve the problem of finding objects which are invariant under the local symmetry. We may think of $G_{N,*}$ as covariant derivatives of the fields ϕ.

There is one subtle point that is sometimes worth remembering if one chooses coset representatives. Although $G_{N,*}$ is naively invariant under g_0 transformations it is not invariant under the required compensating h_c transformation under which $G_{N,*}$ transforms as in eq. (1.2) with h replaced by h_c^{-1}. However, having found a set of dynamics that is invariant under h transformations it is of course also invariant under the compensating transformations.

Realising Yang-Mills theory as a non-linear realisation was first given by Ivanov and Ogievetsky [22] and we now summarise this approach as it will serve as a prototype model for the later sections of this paper. We begin with the algebra

$$P_a\,, \quad J_{ab}\,, \quad Q^{\alpha}, \quad R^{a,\alpha}, \quad K^{a_1a_2,\alpha}, \quad K^{a_1a_2a_3,\alpha}\,, \;\ldots\; K^{a_1\ldots a_n,\alpha} \;\ldots \tag{1.6}$$

which will generate the group G of the non-linear realisation. The generators P_a and J_{ab} are those of the Poincare group while the Q^{α}'s will become identified with those of the gauge group. The generator $R^{a,\alpha}$ is the generator associated to the gauge vector in the non-linear realisation, while the generators $K^{a_1\ldots a_n,\alpha}$ are symmetric in the spacetime indices and will be responsible for the symmetry of the vectors to be promoted to a gauge symmetry. The Q^{α} generators obey the commutators

$$[Q^{\alpha}, Q^{\beta}] = g f^{\alpha\beta}{}_{\gamma}Q^{\gamma}, \tag{1.7}$$

where g is the coupling constant. The remaining commutation relations are given by

$$[K^{a_1\ldots a_n,\alpha}, P_b] = n\delta_b^{(a_1}K^{a_2\ldots a_n),\alpha}\,, \quad [K^{a_1\ldots a_n,\alpha}, K^{b_1\ldots b_m,\beta}] = g f^{\alpha\beta}{}_{\gamma}K^{a_1\ldots a_n b_1\ldots b_m,\gamma}. \tag{1.8}$$

Although the $K^{a_1\ldots a_n,\alpha}$ generators have at least two indices, the commutation relations of Q^{α} and $R^{a,\alpha}$ with all the generators are encoded in the equation above making the identification $K^{a,\alpha} = R^{a,\alpha}$ and $K^{\alpha} = Q^{\alpha}$. The Lorentz generators J_{ab} have the usual commutators with the above generators. The local sub-group H is generated by the Q^{α} and the J_{ab}. As a result we may choose the group element to be of the form

$$g = e^{x^a P_a} \ldots e^{\Phi_{a_1a_2a_3,\alpha}(x)K^{a_1a_2a_3,\alpha}}e^{\Phi_{a_1a_2,\alpha}(x)K^{a_1a_2,\alpha}}e^{A_{a,\alpha}(x)R^{a,\alpha}}. \tag{1.9}$$

Computing the Cartan forms we find that

$$\begin{aligned}
g^{-1}dg &= dx^a[P_a + G_{a,b,\alpha}R^{b,\alpha} + G_{a,bc,\alpha}K^{bc,\alpha} - A_{a,\alpha}Q^\alpha + \cdots] \\
&= dx^a\Bigg[P_a + (\partial_a A_{b,\alpha} - \frac{1}{2}gA_{a,\beta}A_{b,\gamma}f^{\beta\gamma}{}_\alpha - 2\Phi_{ab,\alpha})R^{b,\alpha} \\
&\qquad +\Bigg(\partial_a\Phi_{bc,\alpha} - \frac{1}{6}g^2A_{a,\epsilon}A_{b,\beta}A_{c,\gamma}f^{\epsilon\beta}{}_\delta f^{\delta\gamma}{}_\alpha - 2g\Phi_{ab,\beta}A_{c,\gamma}f^{\beta\gamma}{}_\alpha \\
&\qquad\quad +\frac{1}{2}g\partial_a A_{b,\beta}A_{c,\gamma}f^{\beta\gamma}{}_\alpha - 3\Phi_{abc,\alpha}\Bigg)K^{bc,\alpha} - A_{a,\alpha}Q^\alpha + \cdots\Bigg], \qquad (1.10)
\end{aligned}$$

where the dots denote K generators with more than two spacetime indices. Only the last term in eq. (1.10) is in the local sub-algebra and as such we can identify $A_{a,\alpha}$ as the connection, *i.e.* the gauge field, for the gauge group generated by Q^α. Each of the other terms separately transform covariantly under the local subgroup and so we can place constraints on them and still preserve all the symmetries. In particular we can set

$$G_{(a,b),\alpha} = 0, \qquad (1.11)$$

which implies

$$2\Phi_{ab,\alpha} = \partial_{(a}A_{b),\alpha}, \qquad (1.12)$$

and also

$$G_{(a,bc)\alpha} = 0, \qquad (1.13)$$

which implies

$$3\Phi_{abc,\alpha} = \partial_{(a}\Phi_{bc),\alpha} - 2g\Phi_{(ab,\beta}A_{c),\gamma}f^{\beta\gamma}{}_\alpha + \frac{1}{2}g\partial_{(a}A_{b,\beta}A_{c),\gamma}f^{\beta\gamma}{}_\alpha. \qquad (1.14)$$

Indeed one can solve in this way for all the Φ fields leaving only with the field $A_{a,\alpha}$. The elimination of some fields using constraints on the Cartan forms that preserve the symmetries is sometimes called the inverse Higgs mechanism [23].

Substituting the above solutions for the Φ fields into the Cartan forms one finds expressions that contain $A_{a,\alpha}$ alone which are given by

$$g^{-1}dg = dx^a\left[P_a + F_{ab,\alpha}R^{b,\alpha} + \frac{2}{3}D_bF_{ac,\alpha}K^{bc,\alpha} + \cdots - A_{a,\alpha}Q^\alpha\right], \qquad (1.15)$$

where $F_{ab,\alpha} = \partial_{[a}A_{b]\alpha} - \frac{1}{2}gA_{a,\beta}A_{b,\gamma}f^{\beta\gamma}{}_\alpha$ and D_a is the expected covariant derivative. We recognise this as the Yang-Mills field strength and the higher Cartan form as its covariant derivatives. The object invariant under the symmetries of the non-linear realisation, which is lowest order in derivatives, is just the usual Yang-Mills action.

In fact only the lowest order Cartan form $G_{a,b,\alpha}$ was evaluated in reference [22], but it is interesting to realise that the Cartan forms do contain all the gauge covariant derivatives of the field strength.

One way to arrive at the above set of generators of eq. (1.6) is to write the Yang-Mills gauge parameter as a Taylor expansion

$$\lambda_\alpha(x) = a_\alpha + a_{a,\alpha}x^a + a_{ab,\alpha}x^a x^b + \cdots \qquad (1.16)$$

where the parameters a do not depend on space-time. The usual Yang-Mills transformation can then be interpreted as a an infinite set of rigid transformations whose generators are just those of eq. (1.6) with the commutation relations of eqs. (1.7) and (1.8). Indeed carrying rigid transformations $e^{a\cdot R}$ and $e^{a\cdot K}$ on the group element of eq. (1.9) one finds the same result that a Yang-Mills transformation would produce if the gauge parameter were expanded as in eq. (1.16).

In a tribute to Ogievetsky's important contributions to the theory of non-linear realisations we will call the additional generators $K^{a_1\ldots a_n,\alpha}$ Ogievetsky generators (Og for short) and similarly for their associated fields. They will be used throughout this paper and they are the generators that make the original symmetry, in this case that of the $R^{a,\alpha}$, local. We can systematically assign a grade to the generators, in particular Q^α and P_a have grade -1, $R^{a,\alpha}$ has grade 0 and $K^{a_1\ldots a_{n+1},\alpha}$ have grade n. The coupling constant g has grade -1. We denote the Og generator of grade n as Og n. The algebra of eq. (1.8) can then schematically be written as

$$[G, \text{Og } n] = g \text{ Og } n \quad [\text{Og } n, P_a] = \text{Og } (n-1) \quad [\text{Og } n, \text{Og } m] = g \text{ Og } (m+n+1) \ . \quad (1.17)$$

It will be instructive to consider the dimensional reduction of the above non-linear realisation in D dimensions on a circle with coordinate y. For simplicity we will just consider the abelian case here, and we will therefore drop the index α. After dimensional reduction, the vector field becomes $A_a, A_\star = \varphi$, while the Og 1 field becomes $\Phi_{ab}, \Phi_{a\star}, \Phi_{\star\star}$ and similarly for the higher grade Og fields. Here $\star$ denotes the yth, i.e. circle, components and $a, b = 0, \ldots, D-2$. Neglecting for simplicity the contribution along the Og 1 generator, the Cartan form of eq. (1.10) becomes

$$\begin{aligned} g^{-1}dg = {} & dx^a P_a + dy P_\star + dx^a(\partial_a A_b - 2\phi_{ab})R^b + dx^a(\partial_a\varphi - 2\Phi_{a\star})R^\star \\ & + dy(\partial_\star A_a - 2\Phi_{\star a})R^a + dy(\partial_\star\varphi - 2\Phi_{\star\star})R^\star - dx^a A_a Q - dy\varphi Q. \end{aligned} \quad (1.18)$$

We now take all the fields to be independent of y. Imposing that the Cartan form in the dy direction vanishes, apart from the term in the local subalgebra, we find that $\Phi_{\star a} = \Phi_{\star\star} = 0$. This generalises to all the Og fields of any grade having at least one index in the internal direction. Solving for the remaining Cartan forms as above one finds that one is then left with the fields A_a and φ with the expected dynamics. The net effect of these steps is that from the original set of generators in the higher dimension we take only those that commute with Q, the generator of y transformations, and construct the non-linear realisation from the sub-algebra formed by these generators. Since the Φ fields are related to the derivatives of the usual fields it is to be expected that some of the Ogievetsky fields will vanish in dimensional reduction on a circle.

The non-linear realisation of the Yang-Mills theory will be the prototype example of all the analysis that we will perform throughout this paper. The paper is organised as follows. Section 2 discusses the non-linear realisation of gravity, while section 3 is devoted to the analysis of the 3-form and the 6-form of eleven-dimensional supergravity from E_{11}. In section 4 we show how to derive from E_{11} the Scherk-Schwarz reduction of the IIB theory to nine dimensions. Sections 5 and 6 are devoted to the E_{11} derivation of the massive IIA theory of Romans and of gauged five-dimensional maximal supergravities respectively. In section 7 we discuss the dual graviton in four dimensions, considering first the algebra of the dual graviton alone, and then the cases of gravity and dual gravity in A_1^{+++} in four dimensions and of dual graviton coupled to vectors in E_{11} in four dimensions. Finally, section 8 contains the conclusions.

2 Gravity as a non-linear realisation

It was shown long ago by Borisov and Ogievetsky that four-dimensional gravity could be formulated as a non-linear realisation [3]. These authors showed that gravity in four dimensions could be formulated as the non-linear realisation of $IGL(4,\mathbb{R})$ with local subgroup SO(4) if taken together with the simultaneous realisation of the four dimensional conformal group SO(2, 4) with local subgroup SO(4). The first non-linear realisation possesses coset representatives $g = e^{x\cdot P}e^{h\cdot K}$ that contain the coordinates of spacetime x^μ as coefficients of the spacetime translation generator P_a and the field $h_a{}^b$, which was taken to depend on x^μ, and are associated with the generators $K^a{}_b$ of $GL(4,\mathbb{R})$. The non-linear realisation of the conformal group has coset representatives $g = e^{x\cdot P}e^{\phi D}e^{\phi_a K^a}$ that are labelled by the coordinates of space-time x^μ and the fields ϕ and ϕ_a associated with the dilation generator D and special conformal generator K^a. The field ϕ_a can be eliminated using the inverse Higgs mechanism, that is by setting constraints on the Cartan forms that preserve all the symmetries. The simultaneous non-linear realisation of the two groups is achieved by constructing the dynamics from only the Cartan forms of $IGL(4,\mathbb{R})$ which also transform covariantly under the conformal group. The transformations of the two groups are linked in that the dilation generator D and the trace of the $GL(4,\mathbb{R})$ generators $K^a{}_a$ generate the same scaling of the coordinates x^μ and so their corresponding Goldstone fields ϕ and $h^a{}_a$ must be identified with an appropriate proportionality constant. Although a little complicated the result of this procedure is Einstein's theory if one restricts one's attention to terms that are second order in spacetime derivatives. Taking only the non-linear realisation of $IGL(4,\mathbb{R})$ one can also find Einstein's theory from the Cartan forms provided one fixes a number of coefficients in a way not determined by the symmetries of $IGL(4,\mathbb{R})$ alone. The results can be generalised to D dimensions [2]. However, this latter reference did not use the Lorentz group to make a particular choice of coset representative and introduces a vierbein rather than a metric.

The derivation of gravity as a non-linear realisation was anticipated by an earlier paper of Ogievetsky's [24] that showed that the closure of $IGL(4,\mathbb{R})$ and the conformal group as realised on the coordinates of space-time x^μ in the well known way is equivalent to just considering all infinitesimal general coordinates transformations $x^\mu \to x^\mu + f^\mu(x)$ where $f^\mu(x)$ is an arbitrary function of x^μ. Thus the closure of the two groups is an infinite dimensional group that is just the group of general coordinate transformations. We note that the starting point i.e. the well known transformations on x^μ are just those found by taking spacetime to be a coset or equivalently a non-linear realisation in which the fields are absent.

As such an equivalent more straightforward approach would be take the non-linear realisation of the infinite group which is the closure of the two groups, that is the algebra of general coordinate transformations. This calculation is the subject of this section. Such an approach was adopted by Pashnev [25], however, although we will begin from the same starting point our method will depart in some important ways, some of which are discussed in [26, 27], that are explained below.

Let us begin with the infinite dimensional algebra that contains the generators

$$P_a, K^a{}_b, K^{ab}{}_c, \ldots, K^{a_1\ldots a_n}{}_c, \ldots \tag{2.1}$$

where $K^{a_1\ldots a_n}{}_c = K^{(a_1\ldots a_n)}{}_c$. These generators obey the relations

$$[K^{a_1\ldots a_n}{}_c, P_b] = (n-1)\delta_b^{(a_1} K^{a_2\ldots a_n)}{}_c \tag{2.2}$$

and

$$[K^{a_1\ldots a_n}{}_c, K^{b_1\ldots b_m}{}_d] = (n+m-1)\left[\frac{1}{n}\delta_c^{(b_1|} K^{a_1\ldots a_n|b_2\ldots b_m)}{}_d - \frac{1}{m}\delta_d^{(a_1} K^{a_2\ldots a_n)b_1\ldots b_m}{}_c\right]. \tag{2.3}$$

The generators $P_a, K^a{}_b$ are those of $IGL(D,\mathbb{R})$ while the special conformal transformations are contained in $K^{ab}{}_c$. Indeed the entire algebra can be generated by $P_a, K^a{}_b$ and $K^{ab}{}_c$. We note that one can assign grade to the generators; $K^{a_1\ldots a_{n+1}}{}_c$ has grade n, P_a has grade -1 and $K^a{}_b$ has grade zero. This notion of grade is preserved by the above commutation relations. In terms of our previous notation we call the additional generators Ogievetsky, or Og, generators. In particular, $K^{a_1\ldots a_{n+1}}{}_c$ is an Og n generator. The commutators of eqs. (2.2) and (2.3) can thus schematically be written as

$$[\text{Og } n, \text{Og } m] = \text{Og } (n+m), \tag{2.4}$$

which includes all possible commutators provided that we denote with Og (-1) the momentum operator and with Og 0 the $GL(D,\mathbb{R})$ generators.

We now carry out the non-linear realisation of the group based on the algebra of eqs. (2.2) and (2.3) taking as our local subgroup the Lorentz group which has the generators $J_{ab} = \eta_{[a|c|}K^c{}_{b]}$. As such we may choose our group element, or coset representative to be given by

$$g = e^{x^a P_a} \ldots e^{\Phi^b_{a_1\ldots a_n}(x) K^{a_1\ldots a_n}{}_b} \ldots e^{\Phi^b_{a_1 a_2}(x) K^{a_1 a_2}{}_b} e^{h_a{}^b(x) K^a{}_b} \equiv e^{x^a P_a} g_\phi g_h. \tag{2.5}$$

In fact this is the most general group element as we have not used the Lorentz group to make any choice. The Cartan forms are given by

$$\begin{aligned} g^{-1}dg &= g_h^{-1} g_\phi^{-1} dx^a P_a g_\phi g_h + g_h^{-1}(g_\phi^{-1} dg_\phi) g_h + g_h^{-1} dg_h \\ &= dx^\mu (e_\mu{}^a P_a + G_{\mu,b}{}^c K^b{}_c + G_{\mu,ab}{}^c K^{ab}{}_c + \cdots). \end{aligned} \tag{2.6}$$

A straightforward calculation gives

$$e_\mu{}^a = (e^h)_\mu{}^a, \quad G_{\mu,b}{}^c = (e^{-1}\partial_\mu e)_b{}^c - \Phi_{\mu\rho}{}^\kappa (e^{-1})_b{}^\rho e_\kappa{}^c, \tag{2.7}$$

$$G_{\mu,ab}{}^c = (\partial_\mu \Phi^\lambda_{\rho\kappa} - 2\Phi^\lambda_{\mu\rho\kappa} - \Phi^\tau_{\mu(\rho}\Phi^\lambda_{\kappa)\tau} + \frac{1}{2}\Phi^\tau_{\rho\kappa}\Phi^\lambda_{\mu\tau})(e^{-1})_a{}^\rho (e^{-1})_b{}^\kappa e_\lambda{}^c, \ldots \tag{2.8}$$

In deriving these expressions no conversion of indices on the objects has taken place but the indices have been relabelled with curved or flat indices suitable for their latter interpretation. The factors of e come from the final factor of g_h in the group element. Indeed, carrying out a local Lorentz transformation the Cartan forms transform as in eq. (1.2) and one sees that the $e_\mu{}^a$ are rotated on their a index by a local Lorentz rotation allowing us to interpreted $e_\mu{}^a$ as the vierbein.

The part of the Cartan form involving the local subalgebra is contained in the second term of eq. (2.7) which we may write as

$$G_{\mu,a}{}^b K^a{}_b = G_{\mu,(a}{}^{b)} K^{(a}{}_{b)} + \omega_{\mu a}{}^b J^a{}_b \tag{2.9}$$

where

$$\omega_{\mu a}{}^b = (e^{-1}\partial_\mu e)_{[a}{}^{b]} - \Phi_{\mu\rho}{}^\kappa (e^{-1})_{[a}{}^\rho e_\kappa{}^{b]}. \tag{2.10}$$

We note that although the algebra of eqs. (2.2) and (2.3) is formulated in terms of the generators of $GL(D,\mathbb{R})$ and other generators that are representations of $GL(D,\mathbb{R})$ the choice of the local sub-algebra to be $\mathrm{SO}(1,D-1)$ allows us to introduce the tangent space metric η_{ab} with which we may raise and lower indices to achieve the above (anti-)symmetrisations.

Thus far we agree with the paper of Pashnev [25]. However, in this reference it was proposed that the Maurer Cartan equations $d\mathcal{V} + \mathcal{V}\wedge\mathcal{V} = 0$, which are identities, would place constraints on the fields. Imposing inverse Higgs conditions to find the Christoffel symbol in terms of the metric was correctly carried out in [26, 27].

From now on we follow a different path. The dynamics are constructed in the way explained in the introduction with the Cartan forms transforming as in eq. (1.2). Recalling our discussion in the introduction we conclude that $G_{a,\star} \equiv (e^{-1})_a{}^\mu G_{\mu\star}$, where $\star$ stands for any form except those lying in the Poincare algebra, transform under the Lorentz group as its indices suggest. As such we can place constraints on these Cartan forms and preserve all the symmetries, that is use the inverse Higgs mechanism. Indeed, we can set

$$G_{c,(a}{}^{b)} = (e^{-1})_c{}^\mu G_{\mu(a}{}^{b)} = (e^{-1})_c{}^\mu (e^{-1}\partial_\mu e)_{(a}{}^{b)} - \Phi_{\mu\rho}{}^\kappa (e^{-1})_{(a}{}^\rho e_\kappa{}^{b)} = 0. \tag{2.11}$$

The effect of this is to solve for $\Phi_{\mu\rho}{}^\kappa = \Phi_{(\mu\rho)}{}^\kappa$ in terms of the $e_\mu{}^a$. The result is [26]

$$\Phi_{\mu\nu}{}^\kappa = \Gamma_{\mu\nu}{}^\kappa \equiv \frac{1}{2} g^{\kappa\tau}(\partial_\nu g_{\tau\mu} + \partial_\mu g_{\tau\nu} - \partial_\tau g_{\mu\nu}). \tag{2.12}$$

We define $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$ and recognise $\Gamma_{\mu\nu}{}^\kappa$ as the usual Christoffel connection of general relativity. A quick check of this result is to verify that eq. (2.11) implies that $2g_{\lambda\kappa}\Phi_{\mu\nu}{}^\kappa = \partial_\mu g_{\lambda\nu}$. Substituting into eq. (2.10) we find that

$$\begin{aligned} \omega_{\mu a}{}^b = (e^{-1}\partial_\mu e)_{[a}{}^{b]} - \Gamma_{\mu\rho}{}^\kappa (e^{-1})_{[a}{}^\rho e_\kappa{}^{b]} &= \frac{1}{2} e_a{}^\tau (\partial_\mu e_\tau{}^b - \partial_\tau e_\mu{}^b) \\ -\frac{1}{2}\eta^{bc} e_c{}^\tau (\partial_\mu e_\tau{}^a - \partial_\tau e_\mu{}^a) &- \frac{1}{2} e_a{}^\tau \eta^{bc} e_c{}^\sigma (\partial_\tau e_\sigma{}^d - \partial_\sigma e_\tau{}^d) e_\mu{}^d, \end{aligned} \tag{2.13}$$

which is the well known formula for the spin connection.

At the next level we can covariantly set

$$G_{(d,ab)}{}^c = 0, \tag{2.14}$$

where $G_{d,ab}{}^c \equiv e_d{}^\mu G^c_{\mu,ab} \equiv e_d{}^\mu e_a{}^\rho e_b{}^\kappa G_{\mu,\rho\kappa}{}^\lambda e_\lambda{}^c$. This solves for the field $\Phi_{\mu\nu\rho}{}^\lambda = \Phi_{(\mu\nu\rho)}{}^\lambda$ in terms of $\Phi_{\mu\nu}{}^\lambda$ by imposing symmetrisation in the obvious way. Substituting the solution into the part of this Cartan form that remains we find that

$$2G_{\mu,\rho\kappa}{}^\lambda = R_{\mu\rho}{}^\lambda{}_\kappa \equiv \partial_\mu \Gamma_{\rho\kappa}{}^\lambda - \partial_\rho \Gamma_{\mu\kappa}{}^\lambda + \Gamma_{\mu\tau}{}^\lambda \Gamma_{\rho\kappa}{}^\tau - \Gamma_{\rho\tau}{}^\lambda \Gamma_{\mu\kappa}{}^\tau, \tag{2.15}$$

which we recognise as the well known expression of the Riemann tensor.

At higher orders one imposes covariant constraints on the Cartan forms so as to solve for all the Og fields Φ to leave only the field $h_a{}^b$ or equivalently $e_\mu{}^a = (e^h)_\mu{}^a$. Substituting the solutions back into the Cartan forms we find that

$$g^{-1}dg = dx^\mu \left(e_\mu{}^a P_a + \omega_{\mu a}{}^b J^a{}_b + \frac{1}{2} R_{\mu\rho}{}^\lambda{}_\kappa e_a{}^\rho e_b{}^\kappa e_c{}^\lambda K^{ab}_c + \dots \right), \tag{2.16}$$

where $+\dots$ denotes terms which contain covariant derivatives of the Riemann tensor.

The Og generators play the role of turning $GL(D,\mathbb{R})$ into a local symmetry and one can verify that carrying out a general rigid group transformation $g \to g_0 g$ on the group element of eq. (2.5) we recover the usual general coordinate transformations of general relativity on the vierbein $e_\mu{}^a$.

We will now summarise the above discussion. We started with the group $GL(D,\mathbb{R})$, generators $K^a{}_b$ and the translations P_a to which we assigned grades 0 and -1 respectively. To these we added an infinite number of Ogievetsky generators $K^{a_1\dots a_{n+1}}{}_c$ each with grade n. These obey the Lie algebra of eqs. (2.2) and (2.3). We then placed covariant constraints on the Cartan forms solving for all the Ogievetsky fields whereupon the remaining parts of the Cartan form contain the spin connection at lowest grade and then the Riemann tensor and its covariant derivatives. The introduction of the Ogievetsky generators leads in the non-linear realisation to general coordinate invariance. As such we find Einstein's theory in a completely systematic way from the viewpoint of non-linear realisations.

We now consider the dimensional reduction of this non-linear realisation that is equivalent to the usual dimensional reduction on a circle. Let us denote by y the coordinate of the circle, $*$ the components in this direction and let $Q = P_*$. Dimensionally reducing the Cartan forms of eq. (2.6) we find

$$\begin{aligned} g^{-1}dg = dx^\mu (& e_\mu{}^a P_a + e_\mu{}^* Q + G_{\mu,b}{}^c K^b{}_c + G_{\mu,*}{}^c K^*{}_c + G_{\mu,b}{}^* K^b{}_* + G_{\mu,*}{}^* K^*{}_* \\ & + G_{\mu,ab}{}^c K^{ab}{}_c + 2G_{\mu,*b}{}^c K^{*b}{}_c + \cdots) \\ + dy (& e_*{}^a P_a + e_*{}^* Q + G_{*,b}{}^c K^b{}_c + G_{*,*}{}^c K^*{}_c + G_{*,b}{}^* K^b{}_* + G_{*,*}{}^* K^*{}_* \\ & + G_{*,ab}{}^c K^{ab}{}_c + 2G_{*,*b}{}^c K^{*b}{}_c + G_{*,ab}{}^* K^{ab}{}_* + \cdots). \end{aligned} \tag{2.17}$$

The coefficients G can be read off from eqs. (2.7) and (2.8). We now take all the fields not to depend on y and imposing the inverse Higgs constraint on all $G_{*,\bullet}$ where $\bullet$ is any index, that is set the part of the Cartan form in the dy direction to zero. We find that all the Ogievetsky fields that contain a lower $*$ index vanish. Thus all the Ogievetsky generators that do not commute with Q disappear from the group element and so the Cartan form. The only Ogievetsky fields left are $\Phi_{ab}{}^c$ and Φ^*_{ab}. The later field occurs in the Cartan form in the term

$$dx^\mu G_{\mu,b}{}^* K^b{}_* = ((e^{-1}\partial_\mu e)_b{}^* - \Phi_{\mu\rho}{}^*(e^{-1})_b{}^\rho e_*{}^*) K^b{}_*. \tag{2.18}$$

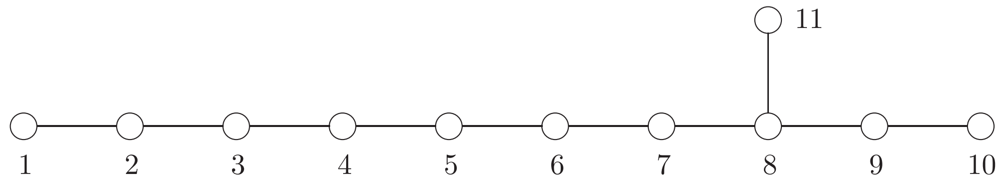

Figure 1. The E_8^{+++}, or E_{11}, Dynkin diagram.

In the dimensionally reduced theory we set the coefficient of dx^μ lying in $K^{(a}{}_{b)}$ of the Cartan form to zero and solve for $\Phi_{ab}{}^c$ which just plays the role of the Ogievetsky field of gravity in the lower dimension. Setting the part of the Cartan form of eq. (2.18) $(e^{-1})_{(a}{}^\mu G_{\mu,b)}{}^* = 0$ we solve for Φ^*_{ab} in terms of $e_\mu{}^*$. The latter field is just the vector field that arises in this dimensional reduction and so this step is as we found for the case of the vector studied earlier. In the dimensionally reduced theory we have as our local symmetry only the Local Lorentz group in the lower dimension. Substituting for $\Phi_{ab}{}^c$ in the part of the Cartan form in this part of the algebra we find the spin connection for the lower dimensional theory. There remains, however, the term containing $(e^{-1})_{[a}{}^\mu G_{\mu,b]}{}^*$, but this we recognise as just the field strength for the vector.

3 E_{11} and eleven-dimensional supergravity

In this section we want to repeat the analysis of the previous section for the non-linear realisation based on the very-extended Kac-Moody algebra E_{11}, whose Dynkin diagram is shown in figure 1.

The decomposition of the adjoint representation of E_{11} with respect to the subalgebra $GL(11,\mathbb{R})$ corresponding to nodes from 1 to 10 in the diagram leads to the generators $K^a{}_b$ of $GL(11,\mathbb{R})$ and R^{abc} and R_{abc} in the completely antisymmetric representations of $GL(11,\mathbb{R})$, together with an infinite set of generators which can be obtained by multiple commutators of the generators R^{abc} and R_{abc} subject to the Serre relations. Defining the level l as the number of times the generator R^{abc} occurs in such multiple commutators, one obtains for instance at level 2 the generator $R^{a_1\dots a_6}$ with completely antisymmetric indices and at level 3 the generator $R^{a,b_1\dots b_8}$ antisymmetric in the indices $b_1\dots b_8$ and with $R^{[a,b_1\dots b_8]} = 0$. The generator R^{abc} itself has level 1, while the generator R_{abc} has level -1 and correspondingly multiple commutators of this generator have negative level [1].

In the last section we have shown how spacetime arises in the nonlinear realisation based on the algebra $GL(D,\mathbb{R})$ in D dimensions. This corresponds to introducing the momentum operator P_a, together with an infinite set of Og n operators $K^{a_1\dots a_{n+1}}{}_b$. In E_{11} the momentum operator arises as the lowest component of the E_{11} representation corresponding to $\lambda_1 = 1$, where λ_1 is the Dynkin index associated to node 1 in figure 1, and called the l multiplet [19]. In this paper we will consider a different approach, that is we will consider the momentum operator as commuting with all the positive level generators. This approach has the advantage that one can naturally introduce the Og operators for

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each positive level generator of E_{11}, although it has the disadvantage of breaking E_{11} to Borel E_{11}, or more precisely to the subgroup of E_{11} generated by $GL(11,\mathbb{R})$ and all the positive level generators. The corresponding local subalgebra is SO(11), or SO(10, 1) in Minkowski signature. We denote with $E_{11}^{\rm local}$ the algebra generated by the momentum operator, the non-negative level E_{11} operators and the Og operators.

In the non-linear realisation, the fields associated to R^{abc} and $R^{a_1\dots a_6}$ correspond to the 3-form and its dual 6-form of eleven dimensional supergravity. The field associated to the generator $R^{a,b_1\dots b_8}$ has the right indices to be associated to the dual graviton, and we will call it the dual graviton for short. In this section we will concentrate on the 3-form and 6-form, while section 7 will be devoted to the dual graviton, although not in eleven dimensions but in the simpler four dimensional case.

Following the analysis of the previous section, we take Og operators for the 3-form and the 6-form in the representations obtained adding symmetrised indices to the set of 3 or 6 antisymmetric indices respectively. The Young tableaux corresponding to the first three Og operators is shown in figure 2. In particular, the Og 1 operators $K_1^{a,b_1b_2b_3}$ and $K_1^{a,b_1\dots b_6}$ belong to the $GL(11,\mathbb{R})$ representations defined by

$$\begin{aligned} &K_1^{a,b_1b_2b_3}=K_1^{a,[b_1b_2b_3]} && K_1^{[a,b_1b_2b_3]}=0\\ &K_1^{a,b_1\dots b_6}=K_1^{a,[b_1\dots b_6]} && K_1^{[a,b_1\dots b_6]}=0 \end{aligned} \tag{3.1}$$

and we take their commutation relation with P_a to be

$$\begin{aligned} [K_1^{a,b_1b_2b_3},P_c]&=\delta^a_c R^{b_1b_2b_3}-\delta_c^{[a}R^{b_1b_2b_3]}\\ [K_1^{a,b_1\dots b_6},P_c]&=\delta^a_c R^{b_1\dots b_6}-\delta_c^{[a}R^{b_1\dots b_6]}. \end{aligned} \tag{3.2}$$

The Og 2 operator for the 3-form $K_2^{a,b,c_1c_2c_3}$ belongs to the representation defined by

$$K_2^{a,b,c_1c_2c_3}=K_2^{(a,b),c_1c_2c_3}=K_2^{a,b,[c_1c_2c_3]}\qquad K_2^{a,[b,c_1c_2c_3]}=0 \tag{3.3}$$

and we take its commutation relation with P_a to be

$$[K_2^{a,b,c_1c_2c_3},P_d]=\delta^a_d K_1^{b,c_1c_2c_3}+\delta^b_d K_1^{a,c_1c_2c_3}+\frac{3}{4}\delta_d^{[c_1}K_1^{|a,b|c_2c_3]}+\frac{3}{4}\delta_d^{[c_1}K_1^{|b,a|c_2c_3]}, \tag{3.4}$$

and similarly for the Og 2 operator for the 6-form $K_2^{a,b,c_1\dots c_6}$. Proceeding this way one can write down the representation and the commutation relation with P_a of the next Og operators. Denoting with n the grade of the Og operators, *i.e.* the Og 1 operators have grade 1, then the commutator of an Og n operator with P_a gives an Og $(n-1)$ operator.

The 6-form generator occurs in the commutator

$$[R^{a_1a_2a_3},R^{a_4a_5a_6}]=2R^{a_1\dots a_6}. \tag{3.5}$$

Using this relation and eq. (3.2) one can then determine the commutation relations between the Og 1 generators and R^{abc} requiring that the Jacobi identities are satisfied. This gives

$$[K_1^{a,b_1b_2b_3},R^{c_1c_2c_3}]=2K_1^{a,b_1b_2b_3c_1c_2c_3}-2K_1^{[a,b_1b_2b_3]c_1c_2c_3}. \tag{3.6}$$

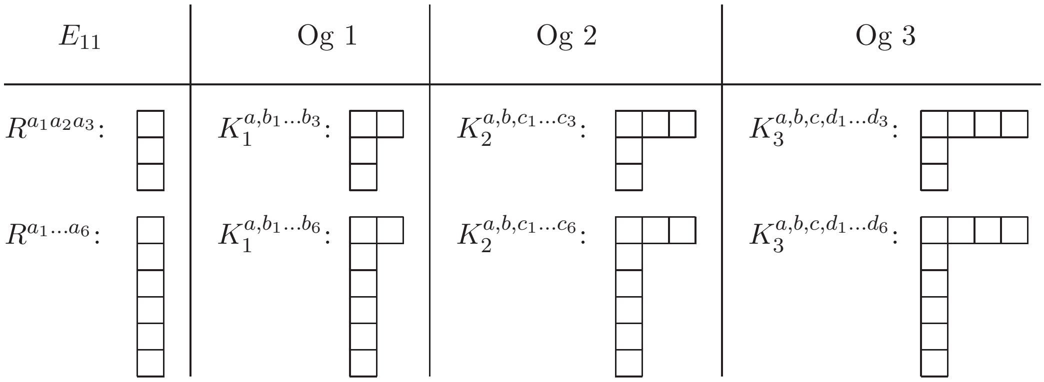

Figure 2. The Young tableaux of the Og generators associated to the eleven-dimensional E_{11} generators R^{abc} and $R^{a_1\ldots a_6}$.

Neglecting higher level generators and the gravity contribution, as well as higher Og generators, we can thus write down the group element as

$$g = e^{x\cdot P} e^{\Phi_{\rm Og} K_1^{\rm Og}} e^{A_{a_1\ldots a_6} R^{a_1\ldots a_6}} e^{A_{a_1\ldots a_3} R^{a_1\ldots a_3}}, \tag{3.7}$$

where we have denoted with $\Phi_{\rm Og}$ the Og 1 fields for both the 3-form and the 6-form, and similarly $K_1^{\rm Og}$ denotes collectively the Og 1 operators for the 3-form and the 6-form. One can then compute the Maurer Cartan form, which is

$$\begin{aligned} g^{-1}\partial_\mu g = {} & P_\mu + (\partial_\mu A_{a_1a_2a_3} - \Phi_{\mu,a_1a_2a_3}) R^{a_1a_2a_3} + (\partial_\mu A_{a_1\ldots a_6} + \partial_\mu A_{a_1a_2a_3} A_{a_4a_5a_6} \\ & - \Phi_{\mu,a_1\ldots a_6} - 2\Phi_{\mu,a_1a_2a_3} A_{a_4a_5a_6}) R^{a_1\ldots a_6} + \cdots \end{aligned} \tag{3.8}$$

The inverse Higgs mechanism allows one to express the Og 1 fields in terms of the 3-form and the 6-form in such a way that only the completely antisymmetric expressions are left in (3.8). This corresponds to

$$\begin{aligned} \Phi_{\mu,a_1a_2a_3} &= \partial_\mu A_{a_1a_2a_3} - \partial_{[\mu} A_{a_1a_2a_3]} \\ \Phi_{\mu,a_1\ldots a_6} &= \partial_\mu A_{a_1\ldots a_6} - \partial_{[\mu} A_{a_1\ldots a_6]} - \partial_\mu A_{a_1a_2a_3} A_{a_4a_5a_6} \\ &\quad - \partial_{[\mu} A_{a_1a_2a_3} A_{a_4a_5a_6]} + 2\partial_{[\mu} A_{a_1a_2a_3]} A_{a_4a_5a_6}, \end{aligned} \tag{3.9}$$

where antisymmetry in the a indices is understood. These relations are all invariant with respect to the local subalgebra. Plugging this into the Maurer-Cartan form one gets

$$g^{-1}\partial_\mu g = P_\mu + F_{\mu a_1a_2a_3} R^{a_1a_2a_3} + F_{\mu a_1\ldots a_6} R^{a_1\ldots a_6} + \cdots, \tag{3.10}$$

where

$$\begin{aligned} F_{a_1a_2a_3a_4} &= \partial_{[a_1} A_{a_2a_3a_4]} \\ F_{a_1\ldots a_7} &= \partial_{[a_1} A_{a_2\ldots a_7]} + F_{[a_1\ldots a_4} A_{a_5a_6a_7]} \end{aligned} \tag{3.11}$$

are the field strengths of the 3-form and its dual 6-form of 11-dimensional supergravity.

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The Maurer-Cartan form is invariant under transformations in the Borel subalgebra, and we use this to derive transformations for the fields. In the particular case discussed in this section, where we have restricted the group element to be as in eq. (3.7), we consider the action of

$$g_0 = e^{a_{a_1a_2a_3}R^{a_1a_2a_3}} e^{a_{a_1\dots a_6}R^{a_1\dots a_6}} e^{b_{a,b_1b_2b_3}K_1^{a,b_1b_2b_3}} e^{b_{a,b_1\dots b_6}K_1^{a,b_1\dots b_6}} \tag{3.12}$$

from the left. Taking the parameters a and b to be infinitesimal, we derive the transformations of the fields to be

$$\begin{aligned}
\delta A_{a_1a_2a_3} &= a_{a_1a_2a_3} + x^b b_{b,a_1a_2a_3} \\
\delta A_{a_1\dots a_6} &= a_{a_1\dots a_6} + a_{a_1a_2a_3}A_{a_4a_5a_6} + x^b b_{b,a_1\dots a_6} + x^b b_{b,a_1a_2a_3}A_{a_4a_5a_6} \\
\delta\Phi_{b,a_1a_2a_3} &= b_{b,a_1a_2a_3} \\
\delta\Phi_{b,a_1\dots a_6} &= b_{b,a_1\dots a_6} - 2\Phi_{b,a_1a_2a_3}a_{a_4a_5a_6} - 2\Phi_{b,a_1a_2a_3}x^c b_{c,a_4a_5a_6}.
\end{aligned} \tag{3.13}$$

The eqs. (3.9) and the field strengths of eqs. (3.11) are separately invariant under these transformations, and in particular the transformations of the 3-form and the 6-form can be written as

$$\begin{aligned}
\delta A_{a_1a_2a_3} &= \partial_{[a_1}\Lambda_{a_2a_3]} \\
\delta A_{a_1\dots a_6} &= \partial_{[a_1}\Lambda_{a_2\dots a_6]} + \partial_{[a_1}\Lambda_{a_2a_3}A_{a_4a_5a_6]}
\end{aligned} \tag{3.14}$$

with gauge parameters

$$\begin{aligned}
\Lambda_{a_1a_2} &= x^b a_{ba_1a_2} + \frac{3}{4}x^b x^c b_{b,ca_1a_2} \\
\Lambda_{a_1\dots a_5} &= x^b a_{ba_1\dots a_5} + \frac{6}{7}x^b x^c b_{b,ca_1\dots a_5}.
\end{aligned} \tag{3.15}$$

Including higher order Og generators corresponds to higher powers of x in the equations above. The full gauge invariance is obtained including all the Og generators.

It is worth mentioning that the normalisation used here is different from the one used in the original E_{11} paper [1]. This is for consistency with the normalisation used in the rest of this paper. Going from this normalisation to the original one in [1] corresponds to making the field redefinitions

$$\begin{aligned}
A_{a_1a_2a_3} &\to \frac{1}{3!}A_{a_1a_2a_3} \\
A_{a_1\dots a_6} &\to \frac{1}{6!}A_{a_1\dots a_6},
\end{aligned} \tag{3.16}$$

as can be deduced from eq. (2.6) in [1].

It is also instructive to consider the Maurer-Cartan form at the next order in the Og generators. For simplicity we will now perform this analysis only for the 3-form, so that we can neglect the contributions coming from commutators of Og generators among themselves. The generalisation to include the 6-form generators is straightforward, although technically more complicated. We thus consider the group element as only containing the

3-form generators in the E_{11} sector and including the Og 1 operator of eq. (3.1) and the Og 2 operator of eq. (3.3), that is

$$g = e^{x\cdot P} e^{\Phi_{a,b,c_1c_2c_3} K_2^{a,b,c_1c_2c_3}} e^{\Phi_{a,b_1b_2b_3} K_1^{a,b_1b_2b_3}} e^{A_{a_1a_2a_3} R^{a_1a_2a_3}} . \tag{3.17}$$

Using eq. (3.4), as well as eq. (3.2), one gets

$$\begin{aligned} g^{-1}\partial_\mu g = {}& P_\mu + (\partial_\mu A_{a_1a_2a_3} - \Phi_{\mu,a_1a_2a_3}) R^{a_1a_2a_3} \\ & + \left(\partial_\mu \Phi_{a,b_1b_2b_3} - \frac{5}{2}\Phi_{\mu,a,b_1b_2b_3}\right) K_1^{a,b_1b_2b_3} + \dots . \end{aligned} \tag{3.18}$$

Using the inverse Higgs mechanism one solves for the Og 1 field in terms of the derivative of the 3-form, and the Og 2 field in terms of the derivative of the Og 1 field. Plugging this into the group element leads to

$$g^{-1}\partial_\mu g = P_\mu + F_{\mu a_1a_2a_3} R^{a_1a_2a_3} + \partial_a F_{\mu b_1b_2b_3} K_1^{a,b_1b_2b_3} + \cdots . \tag{3.19}$$

This is an example of the general picture, in which after applying the inverse Higgs mechanism one is left with the field strengths of the 3-form and the 6-form together with infinitely many derivatives of those, without breaking any of the original symmetries. These fields are the only forms that arise in the decomposition of E_{11} with respect to $GL(11,\mathbb{R})$. Indeed, in [28] it was shown that all the positive level 11-dimensional generators of E_{11} can be cast in generators of the form $R^{9,9,\dots,9,3}$, $R^{9,9,\dots,9,6}$ and $R^{9,9,\dots,9,8,1}$, together with generators with at least one set of 10 or 11 completely antisymmetric indices (here we are using a shortcut notation, in which each number corresponds to the number of antisymmetric indices; for example the Og 2 generator for the 6-form is written as $K_2^{6,1,1}$ in this notation). The fields associated to the former generators (the ones with sets of 9 antisymmetric indices) were interpreted in [28] as being all the possible dual formulations of the 3-form and the graviton, while the latter were interpreted as giving rise to non-propagating fields. In section 7 we will consider the case of E_{11} fields with mixed symmetries, focusing in particular on the case of the dual graviton in four dimensions, while in the next section we will show that the introduction of the Og generators is crucial to understand and derive the algebra that describes gauged supergravity theories.

4 Scherk-Schwarz reduction of IIB supergravity from E_{11}

In this section we will show how to dimensionally reduce maximal supergravities in the context of their E_{11} formulation including the Og extension. We will in particular focus on the case of ten-dimensional IIB reduced to nine dimensions and study both the dimensional reduction on a circle and the Scherk-Schwarz reduction.

We will first introduce the Og generators required to encode the gauge symmetries of the ten-dimensional theory. This gives rise to the algebra $E_{11,10B}^{\text{local}}$. We will then express the E_{11} and Og generators of the IIB theory in a nine-dimensional set-up. The consistency of the truncation from ten-dimensional IIB supergravity to maximal supergravity in nine dimensions corresponds to the fact that within the algebra $E_{11,10B}^{\text{local}}$ of the ten-dimensional

E_{11} and Og generators one can find a sub-algebra $E_{11,9}^{\text{local}}$ appropriate to the nine-dimensional theory. This indeed corresponds to a maximal supergravity theory in nine dimensions, which is a compactification of the ten-dimensional IIB theory on a coordinate y. If one takes the ten-dimensional group element not to depend on y apart from the momentum contribution e^{yQ}, where Q is the internal momentum, then this corresponds to standard, i.e. massless, dimensional reduction on a circle parametrised by y, and the form of the group element is preserved by the sub-algebra $E_{11,9}^{\text{local}}$ of the ten-dimensional algebra $E_{11,10B}^{\text{local}}$ of E_{11} plus Og generators appropriate to massless dimensional reduction. One can also consider a ten-dimensional group element with a suitable y dependence, which we show to give rise in nine dimensions to the massive theory corresponding to the Scherk-Schwarz reduction of the IIB theory [29]. This different form of the group element is preserved by a different sub-algebra of the ten-dimensional algebra $E_{11,10B}^{\text{local}}$ of E_{11} and Og generators that we call $\tilde{E}_{11,9}^{\text{local}}$. We show how to construct this subalgebra corresponding to Scherk-Schwarz reduction. The mass parameter mixes E_{11} and Og generators, and from the nine-dimensional perspective this corresponds to a deformation of the massless E_{11} algebra. The occurrence of a deformed E_{11} algebra associated to massive theories was shown for the first time in [11] for the case of the ten-dimensional massive IIA theory. In that case the occurrence of a mass parameter for the 2-form was shown to arise from requiring that the commutator of the 2-form generator with momentum does not vanish, but is instead equal to the vector generator times the Romans mass parameter.

We now consider the decomposition of the E_{11} generators appropriate to the IIB theory, that arises from deleting node 9 in the Dynkin diagram of figure 1. The $GL(10,\mathbb{R})$ subalgebra associated to the non-linear realisation of gravity corresponds to nodes from 1 to 8 and node 11, while node 10 corresponds to the internal $\mathrm{SL}(2,\mathbb{R})$ symmetry of the IIB theory. We denote tangent spacetime indices in ten dimensions with $\hat{a},\hat{b},\dots$ and curved spacetime indices with $\hat{\mu},\hat{\nu},\dots$, where the indices go from 1 to 10. One constructs the positive level generators as multiple commutators of the 2-form generator $R^{\hat{a}\hat{b},\alpha}$, $\alpha=1,2$, which is a doublet of $\mathrm{SL}(2,\mathbb{R})$. Together with the $GL(10,\mathbb{R})$ generators $K^{\hat{a}}{}_{\hat{b}}$ and the $\mathrm{SL}(2,\mathbb{R})$ generators R^i, $i=1,2,3$ at level zero, one has the doublet of 2-form generators at level 1, a 4-form generator $R^{\hat{a}\hat{b}\hat{c}\hat{d}}$ at level 2, and then a doublet of 6-forms at level 3, a triplet of 8-forms at level 4 and a doublet and a quadruplet of 10-forms at level 5, together with an infinite set of generators with mixed, i.e. not completely antisymmetric, indices. We consider the positive level generators as commuting with the momentum operator $P_{\hat{a}}$.

We now want to write down the relevant algebra in ten dimensions. For simplicity, we consider a level truncation and we therefore only consider in ten dimensions the 2-form generators $R^{\hat{a}\hat{b},\alpha}$, together with the $GL(10,\mathbb{R})$ generators $K^{\hat{a}}{}_{\hat{b}}$ and the $\mathrm{SL}(2,\mathbb{R})$ generators R^i. We have the commutation relations

$$\begin{aligned}[][R^i,R^j] &= f^{ij}{}_k R^k \\ [R^i,R^{\hat{a}\hat{b},\alpha}] &= D^i_\beta{}^\alpha R^{\hat{a}\hat{b},\beta}\end{aligned} \tag{4.1}$$

where $D^i_\beta{}^\alpha$ are the generators of $\mathrm{SL}(2,\mathbb{R})$ satisfying

$$[D^i,D^j]_\beta{}^\alpha = f^{ij}{}_k D^k_\beta{}^\alpha \tag{4.2}$$

and $f^{ij}{}_k$ are the structure constants of SL$(2,\mathbb{R})$. In terms of Pauli matrices, a choice of $D^{i}_{\beta}{}^{\alpha}$ is

$$D_1 = \frac{\sigma_1}{2} \qquad D_2 = \frac{i\sigma_2}{2} \qquad D_3 = \frac{\sigma_3}{2}. \tag{4.3}$$

We now add the Og generators to the E_{11} formulation of ten-dimensional IIB. In this way we encode all the local gauge symmetries of the ten-dimensional IIB theory. The procedure is much like the one discussed in the previous sections for other cases. The Og 1 operator for the 2-form is a doublet of generators $K^{\hat{a},\hat{b}\hat{c},\alpha}$, satisfying

$$K^{\hat{a},\hat{b}\hat{c},\alpha} = K^{\hat{a},[\hat{b}\hat{c}],\alpha} \qquad K^{[\hat{a},\hat{b}\hat{c}],\alpha} = 0, \tag{4.4}$$

and whose commutation relation with the momentum operator $P_{\hat{a}}$ is

$$[K^{\hat{a},\hat{b}\hat{c},\alpha}, P_{\hat{d}}] = \delta^{\hat{a}}_{\hat{d}} R^{\hat{b}\hat{c},\alpha} - \delta^{[\hat{a}}_{\hat{d}} R^{\hat{b}\hat{c}],\alpha}. \tag{4.5}$$

Ignoring for simplicity the gravity contribution, the non-linear realisation can be constructed from the group element

$$g = e^{x\cdot P} e^{\Phi_{\hat{a},\hat{b}\hat{c},\alpha} K^{\hat{a},\hat{b}\hat{c},\alpha}} e^{A_{\hat{a}\hat{b},\alpha} R^{\hat{a}\hat{b},\alpha}} e^{\phi_i R^i} \quad , \tag{4.6}$$

and the corresponding Maurer-Cartan form gives

$$g^{-1}dg = dx^{\hat{\mu}}[P_{\hat{\mu}} + (\partial_{\hat{\mu}} A_{\hat{a}\hat{b},\alpha} - \Phi_{\hat{\mu},\hat{a}\hat{b},\alpha}) e^{-\phi_i R^i} R^{\hat{a}\hat{b},\alpha} e^{\phi_i R^i} + e^{-\phi_i R^i} \partial_{\hat{\mu}} e^{\phi_i R^i} + \cdots] \quad . \tag{4.7}$$

The inverse Higgs mechanism then fixes $\Phi_{\hat{\mu},\hat{a}\hat{b},\alpha}$ in terms of $\partial_{\hat{\mu}} A_{\hat{a}\hat{b},\alpha}$ so that the $R^{\hat{a}\hat{b},\alpha}$ term becomes proportional to

$$F_{\hat{a}\hat{b}\hat{c},\alpha} = \partial_{[\hat{a}} A_{\hat{b}\hat{c}],\alpha} \quad , \tag{4.8}$$

which is the field strength for the 2-form. This procedure is completely consistent because the inverse Higgs mechanism preserves entirely the local subalgebra, which is SO$(9,1) \times$ SO(2).

We now consider a generic compactification of the IIB theory in the above E_{11} formulation to nine dimensions. This will include the derivation of both the massless theory and the Scherk-Schwarz reduction, which both have maximal supersymmetry. We thus split the ten-dimensional coordinates in x^μ, $\mu = 1, \ldots, 9$, and the 10th coordinate y. Correspondingly, the momentum operator splits in P_a and Q, where $Q = P_y$. As we did in ten-dimensions, we consider a level truncation and thus we are only interested in 1-forms and 2-forms in nine dimensions. The doublet of 2-form generators in ten dimensions gives a doublet of 2-forms $R^{ab,\alpha}$ and a doublet of 1-forms $R^{a,\alpha} = R^{ay,\alpha}$. One also obtains a 1-form from the $GL(10,\mathbb{R})$ generators, namely $R^a = K^a{}_y$, whose commutator with P_a is

$$[R^a, P_b] = -\delta^a_b Q. \tag{4.9}$$

One also has the SL$(2,\mathbb{R})$ triplet of scalar generators R^i, as well as the singlet scalar generator $R = K^y{}_y$, satisfying

$$[R, Q] = -Q. \tag{4.10}$$

The commutator between R^a and $R^{a,\alpha}$ is

$$[R^a, R^{b,\alpha}] = -R^{ab,\alpha}, \tag{4.11}$$

while the non-vanishing commutators with the scalars are

$$\begin{aligned} [R, R^a] &= -R^a & [R, R^{a,\alpha}] &= R^{a,\alpha} \\ [R^i, R^{a,\alpha}] &= D^i_\beta{}^\alpha R^{a,\beta} & [R^i, R^{ab,\alpha}] &= D^i_\beta{}^\alpha R^{ab,\beta}. \end{aligned} \tag{4.12}$$

The commutator of R^a with itself and the commutator of $R^{a,\alpha}$ with itself vanish,

$$[R^a, R^b] = 0 \qquad [R^{a,\alpha}, R^{b,\beta}] = 0. \tag{4.13}$$

These are all the E_{11} commutators we need consider at the level we are analysing. At the end of this section we will also consider the 3-form generator R^{abc} and 4-form generator R^{abcd}. The first arises from the 4-form generator of IIB with one index in the internal direction, R^{abcy}, while the latter is just the 4-form of IIB with all indices along the nine-dimensional spacetime.

Just as for the E_{11} generators, we also rewrite the ten-dimensional Og generators as decomposed in $GL(9, \mathbb{R})$ representations. The generator $K^{\hat{a},\hat{b}\hat{c},\alpha}$ thus gives rise to $K^{a,bc,\alpha}$, $K^{[ab],\alpha}$, $K^{(ab),\alpha}$ and $K^{a,\alpha}$, where

$$K^{[ab],\alpha} = K^{y,ab,\alpha} - K^{[a,b]y,\alpha} \qquad K^{(ab),\alpha} = K^{(a,b)y,\alpha} \qquad K^{a,\alpha} = K^{y,ay,\alpha}. \tag{4.14}$$

The commutation relations of these operators with P_a and Q are

$$\begin{aligned} [K^{a,bc,\alpha}, P_d] &= \delta^a_d R^{bc,\alpha} - \delta^{[a}_d R^{bc],\alpha} & [K^{a,bc,\alpha}, Q] &= 0 \\ [K^{[ab],\alpha}, P_c] &= -\delta^{[a}_c R^{b],\alpha} & [K^{[ab],\alpha}, Q] &= R^{ab,\alpha} \\ [K^{(ab),\alpha}, P_c] &= \delta^{(a}_c R^{b),\alpha} & [K^{(ab),\alpha}, Q] &= 0 \\ [K^{a,\alpha}, P_b] &= 0 & [K^{a,\alpha}, Q] &= R^{a,\alpha}. \end{aligned} \tag{4.15}$$

Similarly, the dimensional reduction of the gravity Og 1 operator gives the Og operators $K^{(ab)}$, K^a and K, that satisfy

$$\begin{aligned} [K^{(ab)}, P_c] &= \delta^{(a}_c R^{b)} & [K^{(ab)}, Q] &= 0 \\ [K^a, P_b] &= \delta^a_b R & [K^a, Q] &= R^a \\ [K, P_a] &= 0 & [K, Q] &= R. \end{aligned} \tag{4.16}$$

We now write down the group element. For simplicity we will neglect Og 2 contributions, and therefore we will consider the Og 1 generators as commuting among themselves. We will denote with Φ_{Og} and K^{Og} the whole set of Og 1 fields and generators. Thus the group element is

$$g = e^{x\cdot P} e^{yQ} e^{\Phi_{\mathrm{Og}} K^{\mathrm{Og}}} e^{A_{ab,\alpha} R^{ab,\alpha}} e^{A_{a,\alpha} R^{a,\alpha}} e^{A_a R^a} e^{\phi R} e^{\phi_i R^i}, \tag{4.17}$$

where all the fields are taken to depend on x and y. We now compute the Maurer-Cartan form. The result is

$$\begin{aligned} g^{-1}dg = & dx^\mu[P_\mu + A_\mu e^\phi Q + (\partial_\mu A_{ab,\alpha} - \partial_\mu A_{a,\alpha}A_b - \Phi_{\mu,ab,\alpha} + \Phi_{(\mu a),\alpha}A_b - \Phi_{[\mu a],\alpha}A_b \\ & +\Phi_{(\mu a)}A_{b,\alpha} + \Phi_\mu A_{a,\alpha}A_b)e^{-\phi_i R^i}R^{ab,\alpha}e^{\phi_i R^i} + (\partial_\mu A_{a,\alpha} - \Phi_{(\mu a),\alpha} + \Phi_{[\mu a],\alpha} \\ & -\Phi_\mu A_{a,\alpha})e^{-\phi}e^{-\phi_i R^i}R^{a,\alpha}e^{\phi_i R^i} + (\partial_\mu A_a - \Phi_{(\mu a)} + \Phi_\mu A_a)e^\phi R^a + (\partial_\mu\phi - \Phi_\mu)R \\ & +e^{-\phi_i R^i}\partial_\mu e^{\phi_i R^i} + \cdots] + dy[e^\phi Q + (\partial_y A_{ab,\alpha} - \partial_y A_{a,\alpha}A_b - \Phi_{[ab],\alpha} + \Phi_{a,\alpha}A_b \\ & +\Phi A_{a,\alpha}A_b)e^{-\phi_i R^i}R^{ab,\alpha}e^{\phi_i R^i} + (\partial_y A_{a,\alpha} - \Phi_{a,\alpha} - \Phi A_{a,\alpha})e^{-\phi}e^{-\phi_i R^i}R^{a,\alpha}e^{\phi_i R^i} \\ & +(\partial_y A_a - \Phi_a + \Phi A_a)e^\phi R^a + (\partial_y\phi - \Phi)R + e^{-\phi_i R^i}\partial_y e^{\phi_i R^i} + \dots], \end{aligned} \tag{4.18}$$

where the dots denote contributions from higher level E_{11} generators and Og generators.

Before discussing the Scherk-Schwarz reduction of the IIB theory, we first consider the derivation of the massless nine-dimensional supergravity. We take all the fields in the group element of eq. (4.17) not to depend on y, and using the inverse Higgs mechanism we set the part of the Maurer-Cartan form of eq. (4.18) in the dy direction and proportional to the E_{11} and Og generators to zero. This imposes

$$\Phi_{[ab],\alpha} = \Phi_{a,\alpha} = \Phi_a = \Phi = 0. \tag{4.19}$$

Considering the dx^μ part and imposing the inverse Higgs mechanism on the remaining Og fields one finds that the Maurer-Cartan form gives the field strengths

$$\begin{aligned} F_{abc,\alpha} &= \partial_{[a}A_{bc],\alpha} - \partial_{[a}A_{b,\alpha}A_{c]} \\ F_{ab,\alpha} &= \partial_{[a}A_{b],\alpha} \\ F_{ab} &= \partial_{[a}A_{b]}, \end{aligned} \tag{4.20}$$

which are invariant under the gauge transformations

$$\delta A_{ab,\alpha} = \partial_{[a}\Lambda_{b],\alpha} - \partial_{[a}\Lambda A_{b],\alpha} \qquad \delta A_{a,\alpha} = \partial_a\Lambda_\alpha \qquad \delta A_a = \partial_a\Lambda. \tag{4.21}$$

This construction is consistent because the relations that the inverse Higgs mechanism imposes are invariant under the local subalgebra, which is SO(1,8) $\otimes$ SO(2). The field-strengths and gauge transformations we have derived are those of the 1-forms and 2-forms of massless maximal nine-dimensional supergravity.

In the above, we have set to zero the Og fields corresponding to the generators $K^{[ab],\alpha}$, $K^{a,\alpha}$, K^a and K. Implementing this in the group element, and so the Cartan form, we find that these generators in fact play no role. These generators are indeed the only ones in eqs. (4.15) and (4.16) that do not commute with Q. As such, one is left with the original E_{11} generators and a subset of the Og generators, all of which commute with Q apart from the scalar generator R, and all fields which do not depend on y. We note that the operator Q appears in the commutation relations of eq. (4.9) and (4.10). However, Q commutes with every operator in the theory other that R, and the commutator of R with Q is proportional to Q, and so one can consistently set the commutator of R^a with P_a to zero and ignore

Q in the algebra. Correspondingly, one can ignore the presence of Q in the group element of eq. (4.17), which corresponds to no y dependence at all. Thus one finds a non-linear realisation that is the one that arises if one constructs the massless nine-dimensional theory using the formulation of E_{11} appropriate to nine dimensions, which corresponds to deleting nodes 9 and 11 in the Dynkin diagram in figure 1 and decomposing E_{11} in terms of the $GL(9,\mathbb{R})$ subalgebra. The Og generators that are left are the Og generators that encode the gauge symmetries of the nine-dimensional theory, and they form with the non-negative level E_{11} generators the algebra $E_{11,9}^{\text{local}}$. To summarise, the massless nine-dimensional theory arises from taking the subset of Og generators that commute with Q. This implies that one can consistently remove Q from the algebra, which can be used to construct the non-linear realisation. This in the nine-dimensional E_{11} formulation of massless maximal supergravity.

We now describe the Scherk-Schwarz dimensional reduction of the ten-dimensional IIB supergravity theory to nine dimensions in an analogous way. We take the same starting point, namely the E_{11} formulation of the IIB theory in ten dimensions together with the Og generators and corresponding fields. We consider the group element

$$g = e^{x\cdot P} e^{y(Q+m_i R^i)} e^{\Phi_{\text{Og}}(x) K^{\text{Og}}} e^{A_{ab,\alpha}(x) R^{ab,\alpha}} e^{A_{a,\alpha}(x) R^{a,\alpha}} e^{A_a(x) R^a} e^{\phi(x) R} e^{\phi_i(x) R^i}. \tag{4.22}$$

We thus take the dependence on the coordinate y in the group element to be in the form $e^{y(Q+m_i R^i)}$. This is equivalent to taking the theory to be defined on the conventional nine-dimensional spacetime tensored with a manifold that is a circle constructed form the usual ten-dimensional circle of spacetime and a circle, or one parameter subgroup of $\text{SL}(2,\mathbb{R})$, which is specified by the mass parameter m_i. The factor $e^{y(Q+m_i R^i)}$ occurs at the beginning of the group element in the usual place for the introduction of spacetime, and the fields are taken to not depend on y, however we can rearrange the group element by taking the $e^{y m_i R^i}$ factor to the right whereupon the fields acquire a y dependence, that is

$$g = e^{x\cdot P} e^{yQ} e^{\Phi_{\text{Og}}(x,y) K^{\text{Og}}} e^{A_{ab,\alpha}(x,y) R^{ab,\alpha}} e^{A_{a,\alpha}(x,y) R^{a,\alpha}} e^{A_a(x) R^a} e^{\phi(x) R} e^{y m_i R^i} e^{\phi_i(x) R^i}. \tag{4.23}$$

The y dependence of the fields in this last expression can be derived using the relation

$$e^A e^B e^{-A} = e^{e^A B e^{-A}}. \tag{4.24}$$

This implies in particular that any $\text{SL}(2,\mathbb{R})$ doublet acquires the same y dependence. For instance for the 2-form this is

$$A_{ab,\alpha}(x,y) = (e^{y m_i D^i})_\alpha{}^\beta A_{ab,\beta}(x), \tag{4.25}$$

and similarly for any doublet, including the Og fields, while the $\text{SL}(2,\mathbb{R})$ singlets acquire no y dependence. This is the y dependence that results in the Scherk-Schwarz dimensional reduction, which consists in compactifying the ten-dimensional theory to nine dimensions on a circle of coordinate y, while performing a y-dependent $\text{SL}(2,\mathbb{R})$ transformation [29].

From the group element in (4.23) one obtains the Maurer-Cartan form of eq. (4.18). It is instructive to write this down to show explicitly the y dependence. The result is

$$g^{-1}dg = dx^\mu [P_\mu + A_\mu e^\phi Q + (\partial_\mu A_{ab,\alpha} - \partial_\mu A_{a,\alpha} A_b - \Phi_{\mu,ab,\alpha} + \Phi_{(\mu a),\alpha} A_b - \Phi_{[\mu a],\alpha} A_b \\ + \Phi_{(\mu a)} A_{b,\alpha} + \Phi_\mu A_{a,\alpha} A_b) e^{-\phi_i R^i} e^{-y m_i R^i} R^{ab,\alpha} e^{-y m_i R^i} e^{\phi_i R^i}$$

$$
\begin{aligned}
&+(\partial_\mu A_{a,\alpha} - \Phi_{(\mu a),\alpha} + \Phi_{[\mu a],\alpha} - \Phi_\mu A_{a,\alpha})e^{-\phi}e^{-\phi_i R^i}e^{-ym_i R^i}R^{a,\alpha}e^{ym_i R^i}e^{\phi_i R^i}\\
&+(\partial_\mu A_a - \Phi_{(\mu a)} + \Phi_\mu A_a)e^{\phi}R^a + (\partial_\mu \phi - \Phi_\mu)R + e^{-\phi_i R^i}\partial_\mu e^{\phi_i R^i} + \cdots]\\
&+dy[e^{\phi}Q + (\partial_y A_{ab,\alpha} - \partial_y A_{a,\alpha}A_b - \Phi_{[ab],\alpha} + \Phi_{a,\alpha}A_b\\
&+\Phi A_{a,\alpha}A_b)e^{-\phi_i R^i}e^{-ym_i R^i}R^{ab,\alpha}e^{ym_i R^i}e^{\phi_i R^i} + (\partial_y A_{a,\alpha} - \Phi_{a,\alpha}\\
&-\Phi A_{a,\alpha})e^{-\phi}e^{-\phi_i R^i}e^{-ym_i R^i}R^{a,\alpha}e^{ym_i R^i}e^{\phi_i R^i} + (\partial_y A_a - \Phi_a + \Phi A_a)e^{\phi}R^a\\
&+(\partial_y \phi - \Phi)R + e^{-\phi_i R^i}m_i R^i e^{\phi_i R^i} + \ldots]. \qquad (4.26)
\end{aligned}
$$

Alternatively, one can compute the Maurer-Cartan form with the group element written as in eq. (4.22). In this way of writing down the group element, the fields have no y dependence and the dy part of eq. (4.26) results from passing $m_i R^i$ through the group element. Indeed it can be shown that the two ways of computing the Maurer-Cartan form are identical using the y dependence given as in eq. (4.25).

As for the massless case, we now use the inverse Higgs mechanism to impose that all the terms in dy proportional to positive level generators vanish, and we get

$$
\begin{aligned}
&\Phi_{a,\alpha}(x) = (m_i D^i)_\alpha{}^\beta A_{a,\beta}(x) \qquad\qquad \Phi_{[ab],\alpha}(x) = (m_i D^i)_\alpha{}^\beta A_{ab,\beta}(x)\\
&\quad\Phi = \Phi_a = 0. \qquad (4.27)
\end{aligned}
$$

This does not apply to the scalars ϕ_i, and indeed the $m_i R^i$ term in the dy part of eq. (4.26) is not affected by the inverse Higgs mechanism. This will be discussed later. We put the relations of eq. (4.27) back in the group element and so the Cartan form, and we use the inverse Higgs mechanism on the remaining Og fields such that the dx^μ part of the Cartan form gives the field-strengths

$$
\begin{aligned}
F_{abc,\alpha} &= \partial_{[a}A_{bc],\alpha} - \partial_{[a}A_{b,\alpha}A_{c]} - (m_i D^i)_\alpha{}^\beta A_{[ab,\beta}A_{c]}\\
F_{ab,\alpha} &= \partial_{[a}A_{b],\alpha} + (m_i D^i)_\alpha{}^\beta A_{ab,\beta},\\
F_{ab} &= \partial_{[a}A_{b]}, \qquad (4.28)
\end{aligned}
$$

which transform covariantly under the gauge transformations

$$
\begin{aligned}
\delta A_{ab,\alpha} &= \partial_{[a}\Lambda_{b],\alpha} - \partial_{[a}\Lambda A_{b],\alpha} + \Lambda(m_i D^i)_\alpha{}^\beta A_{ab,\beta}\\
\delta A_{a,\alpha} &= \partial_a \Lambda_\alpha + \Lambda(m_i D^i)_\alpha{}^\beta A_{a,\beta} - (m_i D^i)_\alpha{}^\beta \Lambda_{a,\beta},\\
\delta A_a &= \partial_a \Lambda. \qquad (4.29)
\end{aligned}
$$

These are the field strengths and gauge transformations of the 1-forms and 2-forms of the nine-dimensional gauged maximal supergravity that arises from Scherk-Schwarz reduction of the IIB theory [29].

We now discuss the scalar sector. One obtains the correct covariant derivative for the scalars observing that the metric that results from the Maurer-Cartan form in eq. (4.26) is

$$
\begin{pmatrix} e_\mu{}^a & A_\mu e^{\phi} \\ 0 & e^{\phi} \end{pmatrix} \qquad (4.30)
$$

as the coefficients of the generators P_a and Q (for simplicity we are actually not considering the gravity contribution in nine dimensions and therefore the coefficient of P_a in eq. (4.26) is the diagonal metric). The corresponding inverse metric is

$$\begin{pmatrix} e^{\mu}{}_a & -A_a \\ 0 & e^{-\phi} \end{pmatrix} . \tag{4.31}$$

Taking account of having applied the inverse Higgs mechanism, the only other part of the Maurer-Cartan form along dy is

$$G_{yi} R^i = m_i e^{-\phi_i R^i} R^i e^{\phi_i R^i}, \tag{4.32}$$

while the R^i term along dx^μ is

$$G_{\mu i} R^i = e^{-\phi_i R^i} \partial_\mu e^{\phi_i R^i} \quad . \tag{4.33}$$

Therefore the covariant derivative for the scalar is given by

$$e^{\mu}{}_a G_{\mu,i} - A_a G_{y,i}, \tag{4.34}$$

which reads

$$e^{-\phi_i R^i} \partial_a e^{\phi_i R^i} - A_a m_i e^{-\phi_i R^i} R^i e^{\phi_i R^i}. \tag{4.35}$$

This analysis therefore gives all the covariant quantities of the nine-dimensional theory corresponding to the Scherk-Schwarz reduction of IIB.

As we observed, eq. (4.27) expresses some Og fields in terms of E_{11} fields. Similarly, requiring that the Og 1 operators $K^{[ab],\alpha}$ and $K^{a,\alpha}$ have vanishing coefficients in the dy direction relates $\Phi_{[ab],\alpha}$ and $\Phi_{a,\alpha}$ to Og 2 fields carrying the same spacetime and $\mathrm{SL}(2,\mathbb{R})$ representations. Iterating this one obtains for any n an Og n generator identified with $A_{ab,\alpha}$ times the nth power of the mass parameter, and similarly for $A_{a,\alpha}$. This generalises to all the fields in the theory. Putting these solutions into the original group element of eq. (4.22) we find that it takes the form

$$g = e^{x \cdot P} e^{y(Q + m_i R^i)} e^{\Phi_{\rm Og}(x) \tilde{K}^{\rm Og}} e^{A_{ab,\alpha}(x) \tilde{R}^{ab,\alpha}} e^{A_{a,\alpha}(x) \tilde{R}^{a,\alpha}} e^{A_a(x) R^a} e^{\phi(x) R} e^{\phi_i(x) R^i}, \tag{4.36}$$

where

$$\begin{aligned} \tilde{R}^{a,\alpha} &= R^{a,\alpha} + m_i D^i_\beta{}^\alpha K^{a,\beta} + \cdots \\ \tilde{R}^{ab,\alpha} &= R^{ab,\alpha} + m_i D^i_\beta{}^\alpha K^{[ab],\beta} + \cdots , \end{aligned} \tag{4.37}$$

where the dots correspond to higher powers in m_i multiplying higher grade Og generators, and $\tilde{K}$ denotes deformed Og generators associated with nine-dimensional gauge transformations. The group element of eq. (4.36) resembles the group element corresponding to the massless nine-dimensional theory, in the sense that each generator in eq. (4.36) corresponds to a generator with identical index structure of the massless nine-dimensional theory. As such we can interpret the $\tilde{R}$ generators as deformed E_{11} generators. In particular, we claim that although the expansions in eq. (4.37) are non-polynomial in m_i, all the commutation relations involving these operators, or the commutation relations between these operators and momentum, only contain terms at most linear in m_i. In particular, the commutator of $\tilde{R}^{ab,\alpha}$ with P_a is

$$[\tilde{R}^{ab,\alpha}, P_c] = -(m_i D^i)_\beta{}^\alpha \delta_c^{[a} \tilde{R}^{b],\beta}, \tag{4.38}$$

while the commutator of $\tilde{R}^{a,\alpha}$ with P_a vanishes, as can be seen from eq. (4.15).

The deformed E_{11} and indeed the deformed Og generators have a simple algebraic classification. They are the operators that commute with the operator $\tilde{Q}$ defined as

$$\tilde{Q} = Q + m_i R^i. \tag{4.39}$$

Indeed, the commutation relation of R^a, $R^{a,\alpha}$ and $R^{ab,\alpha}$ with $\tilde{Q}$ is

$$\begin{aligned}
[\tilde{Q}, R^a] &= 0 \\
[\tilde{Q}, R^{a,\alpha}] &= m_i D^i_\beta{}^\alpha R^{a,\beta} \\
[\tilde{Q}, R^{ab,\alpha}] &= m_i D^i_\beta{}^\alpha R^{ab,\beta}.
\end{aligned} \tag{4.40}$$

We thus have to deform the operators $R^{a,\alpha}$ and $R^{ab,\alpha}$, and from eq. (4.15) one gets that the deformed operators given in eq. (4.37) satisfy

$$[\tilde{Q}, \tilde{R}^{a,\alpha}] = [\tilde{Q}, \tilde{R}^{ab,\alpha}] = 0. \tag{4.41}$$

The Og generators of the nine-dimensional theory are also redefined in order to commute with $\tilde{Q}$. One thus constructs the generators $\tilde{K}^{a,bc,\alpha}$ and $\tilde{K}^{(ab),\alpha}$, which are Og 1 generators followed by an expansion in m_i of higher grade Og generators. The Og 1 generator $K^{(ab)}$ of the singlet vector R^a is not modified as it commutes with $\tilde{Q}$.

Having introduced the operator $\tilde{Q}$, we can write down the group element of eq. (4.36) as

$$g = e^{x\cdot P} e^{y\tilde{Q}} e^{\Phi_{\rm Og}(x)\tilde{K}^{\rm Og}} e^{A_{ab,\alpha}(x)\tilde{R}^{ab,\alpha}} e^{A_{a,\alpha}(x)\tilde{R}^{a,\alpha}} e^{A_a(x)R^a} e^{\phi(x)R} e^{\phi_i(x)R^i}. \tag{4.42}$$

Indeed, we now show that using the operator $\tilde{Q}$ rather than Q one obtains the field strengths, including the covariant derivative of the scalars, in a straightforward way. Calculating the Cartan forms from the group element of eq. (4.42) and using eq. (4.38) and the fact that all the positive level operators commute with $\tilde{Q}$ we find

$$\begin{aligned}
g^{-1}dg = dx^a[&P_a + (\partial_a A_{bc,\alpha} - \partial_a A_{b,\alpha} A_c - (m_i D^i)_\alpha{}^\beta A_{ab,\beta} A_c + \cdots) e^{-\phi_i R^i} \tilde{R}^{bc,\alpha} e^{\phi_i R^i} \\
&+(\partial_a A_{b,\alpha} + (m_i D^i)_\alpha{}^\beta A_{ab,\beta} + \cdots) e^{-\phi} e^{-\phi_i R^i} \tilde{R}^{b,\alpha} e^{\phi_i R^i} + (\partial_a A_b + \cdots) e^{\phi} R^b \\
&+\partial_a \phi R + A_a e^{-\phi_i R^i} e^{-\phi R} \tilde{Q} e^{\phi R} e^{\phi_i R^i} + e^{-\phi_i R^i}(\partial_a - A_a m_i R^i) e^{\phi_i R^i}] \\
&+dy e^{-\phi_i R^i} e^{-\phi R} \tilde{Q} e^{\phi R} e^{\phi_i R^i},
\end{aligned} \tag{4.43}$$

where the dots in each term denote the Og field contributions, whose role is to cancel the non-antisymmetric terms in the Cartan form using the inverse Higgs mechanism. As explained above $g^{-1}dg$ is invariant under $g \to g_0 g$ and so all the coefficients of the generators in the above equation are invariant. Hence, in particular the two terms

$$dx^a P_a, \qquad dx^a e^{-\phi_i R^i}(\partial_a - A_a m_i R^i) e^{\phi_i R^i} \tag{4.44}$$

are separately invariant under g_0 transformations. Hence we can identify the covariant derivative of the scalars as

$$e^{-\phi_i R^i}(\partial_a - A_a m_i R^i)e^{\phi_i R^i} \tag{4.45}$$

which now only transforms under the local transformations. The infinite number of rigid g_0 transformations constitute the gauge transformations and so this covariant derivative is also covariant in the conventional sense.

We note that the operator $\tilde{Q}$ and the variable y although important for the logic of the result did not appear explicitly in the calculation of the terms that lead to this covariant derivative. Indeed one could have written down the group element without any $\tilde{Q}$ or y dependence and the Cartan forms would give the correct covariant derivatives and so gauge invariant quantities. Dropping the operator $\tilde{Q}$, one obtains in particular the commutation relation

$$[R^a, P_b] = \delta^a_b m_i R^i \quad . \tag{4.46}$$

We now consider eq. (4.46) as our starting point to define the nine-dimensional algebra $\tilde{E}^{\text{local}}_{11,9}$. This is the algebra that describes the deformed nine-dimensional theory considered in this section, and contains the generators $m_i R^i$, P_a and all the positive level deformed generators, including the deformed Og generators. In the remaining of this section we will show that all the results obtained so far can be derived simply requiring the closure of the Jacobi identities in $\tilde{E}^{\text{local}}_{11,9}$ starting from eq. (4.46). This approach is entirely nine-dimensional, and one never makes use of the fact that the theory has a ten-dimensional origin. As we will see, this provides an extremely fast method of deriving the field strengths of all gauged maximal supergravities.

We start considering the Jacobi identity involving R^a, $\tilde{R}^{a,\alpha}$ and P_a. The commutator between R^a and $\tilde{R}^{b,\alpha}$ is a deformation of the commutator in eq. (4.11), and the most general expression we can write with the generators at our disposal is

$$[R^a, \tilde{R}^{b,\alpha}] = -\tilde{R}^{ab,\alpha} + a m_i D^i_\beta{}^\alpha \tilde{K}^{(ab),\alpha} \quad , \tag{4.47}$$

with a to be determined, and where $\tilde{K}^{(ab),\alpha}$ is the modified Og 1 generator satisfying

$$[\tilde{K}^{(ab),\alpha}, P_c] = \delta^{(a}_c \tilde{R}^{b),\alpha}. \tag{4.48}$$

We also demand that the commutator between $\tilde{R}^{ab,\alpha}$ and P_c be of the form

$$[\tilde{R}^{ab,\alpha}, P_c] = b(m_i D^i)_\beta{}^\alpha \delta^{[a}_c \tilde{R}^{b],\beta}, \tag{4.49}$$

with the parameter b to be determined. The Jacobi identity involving R^a, $\tilde{R}^{a,\alpha}$ and P_a is satisfied provided that the values of a and b are

$$a = 1 \qquad\qquad b = -1. \tag{4.50}$$

To summarise, we have obtained the relations

$$\begin{aligned}
[R^a, \tilde{R}^{b,\alpha}] &= -\tilde{R}^{ab,\alpha} + m_i D^i_\beta{}^\alpha \tilde{K}^{(ab),\alpha} \\
[\tilde{R}^{ab,\alpha}, P_c] &= -(m_i D^i)_\beta{}^\alpha \delta^{[a}_c \tilde{R}^{b],\beta},
\end{aligned} \tag{4.51}$$

and in particular the second relation coincides with eq. (4.38).

Proceeding this way, one can determine all the commutation relations of the modified E_{11} generators among themselves and with the momentum operator P_a. For instance, the Jacobi identity involving the operators R^a, $\tilde{R}^{ab,\alpha}$ and P_a requires the cancellation of terms linear in m_i as well as terms quadratic in m_i. The latter are cancelled by requiring that also the commutator of $\tilde{K}^{a,bc,\alpha}$ with P_d receives a correction at order m_i. The result is

$$[R^a, \tilde{R}^{bc,\alpha}] = \frac{3}{2}(m_i D^i)_\beta{}^\alpha \tilde{K}^{a,bc,\beta} \tag{4.52}$$

and

$$[\tilde{K}^{a,bc,\alpha}, P_d] = \delta^a_d \tilde{R}^{bc,\alpha} - \delta^{[a}_d \tilde{R}^{bc],\alpha} - \frac{1}{3}(m_i D^i)_\beta{}^\alpha (\delta^b_d \tilde{K}^{ac,\beta} - \delta^c_d \tilde{K}^{ab,\beta}). \tag{4.53}$$

Using the definition of the operator $\tilde{R}^{ab,\alpha}$ in eq. (4.37), and eq. (4.14), one can for instance recover eq. (4.52), that we have obtained requiring the closure of the Jacobi identities, directly using the ten-dimensional commutation relations. Indeed, at lowest order in the mass parameter, one gets

$$[R^a, \tilde{R}^{bc,\alpha}] = (m_i D^i)_\beta{}^\alpha [K^a{}_y, K^{y,bc,\beta} - K^{[b,c]y,\beta}] = \frac{3}{2}(m_i D^i)_\beta{}^\alpha K^{a,bc,\beta}. \tag{4.54}$$

In order to show the power of this method, we now determine the field-strengths for the 3-form and the 4-form of the nine-dimensional massive theory without using its ten-dimensional origin. We first write down the relevant commutators of the massless theory. We thus add to the commutators of eqs. (4.11) and (4.12) all the commutators that involve generators up to the 4-form included. We only write down the non-vanishing commutators, that are

$$\begin{aligned} &[R, R^{abc}] = R^{abc} && [R^a, R^{bcd}] = -R^{abcd} \\ &[R^{a,\alpha}, R^{bc,\beta}] = \epsilon^{\alpha\beta} R^{abc} && [R^{ab,\alpha}, R^{cd,\beta}] = \epsilon^{\alpha\beta} R^{abcd}, \end{aligned} \tag{4.55}$$

where $\epsilon^{\alpha\beta}$ is the invariant antisymmetric tensor of $\mathrm{SL}(2,\mathbb{R})$. Starting from these relations and using eq. (4.46) we can determine all the commutation relations involving such deformed generators by imposing the closure of the Jacobi identities. Denoting with $\tilde{R}^{abc}$ and $\tilde{R}^{abcd}$ the deformed generators, one can show that the only commutation relation that needs to be modified with respect to eq. (4.55) is the commutator between two deformed 2-form generators, which becomes

$$[\tilde{R}^{ab,\alpha}, \tilde{R}^{cd,\beta}] = \epsilon^{\alpha\beta} \tilde{R}^{abcd} + 2(m_i D^i)^{\alpha\beta} \tilde{K}^{[a,b]cd}, \tag{4.56}$$

where $\tilde{K}^{a,bcd}$ is the deformed Og 1 generator associated to the deformed 3-form generator $\tilde{R}^{abc}$, satisfying

$$[\tilde{K}^{a,b_1b_2b_3}, P_c] = \delta^a_c \tilde{R}^{b_1b_2b_3} - \delta^{[a}_c \tilde{R}^{b_1b_2b_3]} \tag{4.57}$$

and we have used $\epsilon^{\alpha\beta}$ to raise the $\mathrm{SL}(2,\mathbb{R})$ index, that is

$$D^{i\ \alpha\beta} = \epsilon^{\alpha\gamma} D^i_\gamma{}^\beta, \tag{4.58}$$

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and $D^{i\ \alpha\beta}$ is symmetric. The deformed Og 1 generator for the 4-form is $\tilde{K}^{a,b_1b_2b_3}$, satisfying

$$[\tilde{K}^{a,b_1\ldots b_4}, P_c] = \delta^a_c \tilde{R}^{b_1\ldots b_4} - \delta^{[a}_c \tilde{R}^{b_1\ldots b_4]}. \tag{4.59}$$

Both the deformed 3-form and the deformed 4-form commute with the momentum operator (actually neither the 3-form nor the 4-form generator are really deformed, but this is not relevant for this analysis).

We now consider the group element

$$g = e^{x\cdot P} e^{\Phi_{\rm Og}\tilde{K}^{\rm Og}} e^{A_{abcd}\tilde{R}^{abcd}} e^{A_{abc}\tilde{R}^{abc}} e^{A_{ab,\alpha}\tilde{R}^{ab,\alpha}} e^{A_{a,\alpha}\tilde{R}^{a,\alpha}} e^{A_a R^a} e^{\phi R} e^{\phi_i R^i}, \tag{4.60}$$

which only depends on the nine-dimensional coordinates x^a. Computing the Maurer-Cartan form and applying the inverse Higgs mechanism, one can show that all the terms which are not antisymmetric are set to zero by fixing the Og 1 fields in terms of the E_{11} fields, and one is left with completely antisymmetric terms. These are the field-strengths of the 1-forms and 2-forms given in eq. (4.28), as well as the field-strengths

$$\begin{aligned} F_{a_1\ldots a_4} &= \partial_{[a_1}A_{a_2a_3a_4]} + \epsilon^{\alpha\beta}\partial_{[a_1}A_{a_2a_3,\alpha}A_{a_4],\beta} - \frac{1}{2}(m_iD^i)^{\alpha\beta}A_{[a_1a_2,\alpha}A_{a_3a_4],\beta} \\ F_{a_1\ldots a_5} &= \partial_{[a_1}A_{a_2\ldots a_5]} - \partial_{[a_1}A_{a_2a_3a_4}A_{a_5]} + \frac{1}{2}\epsilon^{\alpha\beta}\partial_{[a_1}A_{a_2a_3,\alpha}A_{a_4a_5],\beta} \\ &\quad -\epsilon^{\alpha\beta}\partial_{[a_1}A_{a_2a_3,\alpha}A_{a_4,\beta}A_{a_5]} + \frac{1}{2}(m_iD^i)^{\alpha\beta}A_{[a_1a_2,\alpha}A_{a_3a_4,\beta}A_{a_5]} \end{aligned} \tag{4.61}$$

for the 3-form and the 4-form. These are indeed the field-strengths of the 3-form and its dual 4-form of the massive nine-dimensional supergravity. The gauge transformations of the fields arise in the non-linear realisation as rigid transformations of the group element with the Og generators included. One obtains the transformations of the 2-forms and 1-forms given in eq. (4.29) as well as the transformations

$$\begin{aligned} \delta A_{abc} &= \partial_{[a}\Lambda_{bd]} + \epsilon^{\alpha\beta}\partial_{[a}\Lambda_\alpha A_{bc],\beta} + \frac{1}{2}\epsilon^{\alpha\beta}\partial_{[a}\Lambda A_{b,\alpha}A_{c],\beta} + (m_iD^i)^{\alpha\beta}\Lambda_{[a,\alpha}A_{bc],\beta} \\ \delta A_{a_1\ldots a_4} &= \partial_{[a_1}\Lambda_{a_2a_3a_4]} + \frac{1}{2}\epsilon^{\alpha\beta}\partial_{[a_1}\Lambda_{a_2,\alpha}A_{a_3a_4],\beta} - \partial_{[a_1}\Lambda A_{a_2a_3a_4]} \\ &\quad + \frac{1}{2}\epsilon^{\alpha\beta}\partial_{[a_1}\Lambda A_{a_2,\alpha}A_{a_3a_4],\beta} \end{aligned} \tag{4.62}$$

of the 3-form and the 4-form.

Observe that although the operators R^i other than m_iR^i do not belong to the algebra $\tilde{E}^{\rm local}_{11,9}$, one can nonetheless use the group element of eq. (4.60). Indeed, the covariant derivative for the scalars is also obtained from the nine-dimensional group element of eq. (4.60). Indeed, the Maurer-Cartan form contains the terms

$$e^{-\phi_iR^i}\partial_\mu e^{\phi_iR^i} - A_\mu e^{-\phi_iR^i}m_iR^i e^{\phi_iR^i}, \tag{4.63}$$

which we recognise as the covariant derivative of the scalars.

To summarise, we have found a general pattern for carrying out dimensional reduction to obtain gauged supergravities. The higher dimensional coordinates have a generator (Q

in the massless case and $\tilde{Q}$ in the Scherk-Schwarz case) which is associated with the space being reduced on. From the set on E_{11} and Og generators, we can find a set of deformed E_{11} generators which are just those that commute with the preferred generator associated with the reduction (Q or $\tilde{Q}$). The field strengths can then just be deduced from this deformed E_{11} algebra. We will see that this method transcends dimensional reduction and in fact applies to all gauged maximal supergravities. This will be the focus on the next two sections.

5 E_{11} and massive IIA

In the last section we have analysed the massless and Scherk-Schwarz reductions to nine dimensions of ten-dimensional IIB supergravity from an E_{11} perspective. Starting from the algebra $E^{\text{local}}_{11,10B}$ of E_{11} plus the Og generators that encodes all the gauge symmetries of the ten-dimensional theory, the massless dimensional reduction corresponds to taking the E_{11} generators together with the subset of Og generators that commute with the momentum operator in the internal direction. On the other hand, the Scherk-Schwarz dimensional reduction corresponds to choosing operators that commute with a twisted internal momentum operator, and the twist is such that these operators are combinations of the ten-dimensional E_{11} and Og generators. It is important to stress that the content of the sets of generators in the massless and the Scherk-Schwarz theory are exactly the same, and the two theories differ because the commutation relations are different. In particular, the set of non-negative level E_{11} generators in the massless theory is the same as the set of operators in the Scherk-Schwarz reduction case that are obtained by adding to the E_{11} generators suitable Og generators of the ten-dimensional theory multiplied by powers of the mass deformation parameter m_i, where i is an $\mathrm{SL}(2,\mathbb{R})$ triplet index. From the nine-dimensional perspective, these operators look like E_{11} generators, but their commutation relation receives a correction at order m_i. Therefore, the algebra appears from the nine-dimensional perspective as a deformation of the original E_{11} algebra. This deformation is such that the commutator of two positive level generators gives the standard E_{11} result at zero order in m_i together with an order m_i deformation proportional to the Og generators of the nine-dimensional theory. Correspondingly, the commutator of the deformed positive level E_{11} generators with the nine-dimensional momentum is proportional to the deformed E_{11} generators times the mass parameter. Starting from the commutation relation (4.46), the entire algebra of the nine-dimensional deformed theory can be determined by requiring that the Jacobi identities close.

In this section we consider the case of the massive deformation of the IIA theory, discovered by Romans in [8]. In this case the theory does not arise as a dimensional reduction of eleven-dimensional supergravity, and therefore one cannot deform the E_{11} generators adding eleven-dimensional Og generators. Nonetheless, we will show that from the ten-dimensional perspective one can still consider deformed E_{11} and Og generators, and the corresponding algebra $\tilde{E}^{\text{local}}_{11,10A}$, which appears as a deformation of the massless ten-dimensional algebra, determines all the field-strengths of the theory. In [11] it was shown that the massive IIA theory can be recovered from an E_{11} perspective by adopting a non-trivial commutation relation between the momentum operator and the positive level generators. In particular the commutator of the 2-form generator with momentum gives the 1-form generator multiplied by the mass deformation parameter. The resulting algebra [11] though has a problem of consistency because the corresponding Jacobi identities do not close. In [30] it was shown that if one insists on requiring the consistency of the algebra for the lower-rank forms, the commutator of two 2-forms cannot vanish in the massive theory, but instead is proportional to an operator in the (3,1) representation of $GL(10,\mathbb{R})$. This operator is indeed the Og 1 operator for the 3-form. We show that the whole algebra corresponding to the massive IIA theory is determined starting from the deformed commutation relation of the 2-form with momentum and requiring the closure of all the Jacobi identities. A different approach, based on the Kac-Moody algebra E_{10} [31], has recently been given in [32].

We start by writing down the algebra associated to the massless IIA theory. The massless IIA theory arises from the dimensional reduction of eleven-dimensional supergravity. The corresponding algebra arises from a decomposition of the E_{11} algebra in terms of $GL(10,\mathbb{R})$ as relevant for the IIA theory, which corresponds to deleting nodes 10 and 11 in the Dynkin diagram in figure 1. In this section we denote with $a, b, \ldots$ the tangent spacetime indices in ten dimensions. In deriving the E_{11} generators in terms of their $GL(10,\mathbb{R})$ IIA representations it is useful to consider the eleven-dimensional generators and denote with y the internal 11th coordinate. One then obtains that the theory contains a scalar R, which is the $GL(11,\mathbb{R})$ generator $K^y{}_y$, a vector R^a corresponding to the eleven-dimensional $K^a{}_y$, a 2-form R^{ab} which arises from the eleven-dimensional 3-form with one index in the internal direction R^{aby}, and then a 3-form R^{abc}, a 5-form $R^{a_1\ldots a_5}$, a 6-form $R^{a_1\ldots a_6}$, a 7-form $R^{a_1\ldots a_7}$, an 8-form $R^{a_1\ldots a_8}$, a 9-form $R^{a_1\ldots a_9}$ and two 10-forms $R^{a_1\ldots a_{10}}$ and $R'^{a_1\ldots a_{10}}$, together with an infinite set of generators with mixed, i.e. not completely antisymmetric, indices [12].

We now write down the part of the E_{11} algebra that involves these completely antisymmetric generators. This was first derived in [11], but we use different normalisations for the generators, that make the eleven-dimensional origin of the algebra more transparent. For simplicity, we will neglect the contribution from the 10-form generators, that is we will consider a level truncation only involving generators up to the 9-form included. The algebra is

$$\begin{aligned}
&[R, R^a] = -R^a && [R, R^{ab}] = R^{ab} \\
&[R, R^{a_1\ldots a_5}] = R^{a_1\ldots a_5} && [R, R^{a_1\ldots a_7}] = 2R^{a_1\ldots a_7} \\
&[R, R^{a_1\ldots a_8}] = R^{a_1\ldots a_8} && [R, R^{a_1\ldots a_9}] = 3R^{a_1\ldots a_9} \\
&[R^a, R^{bc}] = R^{abc} && [R^{a_1}, R^{a_2\ldots a_6}] = -R^{a_1\ldots a_6} \\
&[R^{a_1}, R^{a_2\ldots a_8}] = 3R^{a_1\ldots a_8} && [R^{a_1a_2}, R^{a_3\ldots a_5}] = -2R^{a_1\ldots a_5} \\
&[R^{a_1a_2}, R^{a_3\ldots a_7}] = R^{a_1\ldots a_7} && [R^{a_1a_2}, R^{a_3\ldots a_8}] = -2R^{a_1\ldots a_8} \\
&[R^{a_1a_2}, R^{a_3\ldots a_9}] = R^{a_1\ldots a_9} && [R^{a_1\ldots a_3}, R^{a_4\ldots a_6}] = 2R^{a_1\ldots a_6} \\
&[R^{a_1\ldots a_3}, R^{a_4\ldots a_8}] = R^{a_1\ldots a_8}, &&
\end{aligned} \tag{5.1}$$

with all the other commutators vanishing or giving generators with mixed symmetries. One can show that all the Jacobi identities involving these operators are satisfied. Following the results of section 4, one can then obtain the Og generators of the ten-dimensional theory by decomposing the Og generators of the eleven-dimensional theory in terms of representations of $GL(10,\mathbb{R})$. The subset of such generators that commute with the momentum operator along the 11th direction are the Og generators of the massless IIA theory. These are exactly the operators that are needed to encode all the gauge symmetries of the fields of the massless IIA theory. This corresponds to the fact that the massless IIA theory arises as a circle dimensional reduction of eleven-dimensional supergravity. In particular, for each n-form E_{11} generator the corresponding Og 1 operator is $K^{a,b_1\ldots b_n}$ satisfying $K^{[a,b_1\ldots b_n]}=0$.

We now consider the deformation of the massless IIA algebra giving rise to massive IIA. This theory was constructed by Romans in [8], and it corresponds to a Higgs mechanism in which the 2-form acquires a mass by absorbing the vector. In [11] this mechanism was recovered from an E_{11} perspective by adopting a non-trivial commutation relation between the 2-form generator and momentum. Following the results of section 4, we interpret this as a redefinition of the E_{11} generators. We thus denote all the generators of the massive theory with a tilde. These generators, although forming a set identical to the one corresponding to the E_{11} generators of the massless theory, have different commutation relations. These commutation relations make the corresponding algebra look like a deformation of the massless algebra involving the mass parameter. We thus write down the commutation relation between the 2-form and momentum as

$$[\tilde{R}^{ab},P_c]=-m\delta_c^{[a}\tilde{R}^{b]}, \tag{5.2}$$

where m is the Romans mass parameter. Our strategy is to use eq. (5.2) as our starting point, and to derive all the commutation relations of the deformed theory from it imposing the closure of the Jacobi identities. We will show that this will fix all the field-strengths and gauge transformations of the forms in the theory. In [33] it was shown that the supersymmetry algebra of IIA closes on all the forms predicted by E_{11}, and the field-strengths and gauge transformations of the form fields were derived imposing the closure of the supersymmetry algebra. We will show that the field-strengths and gauge transformations as obtained using supersymmetry exactly coincide, up to field redefinitions, with the ones obtained here from E_{11}.

The Jacobi identity involving the operators $\tilde{R}$, $\tilde{R}^{ab}$ and P_a imposes that

$$[\tilde{R},P_a]=-2P_a. \tag{5.3}$$

Introducing the Og 1 operator for the deformed 3-form, defined as

$$[\tilde{K}^{a,b_1b_2b_3},P_c]=\delta_c^a\tilde{R}^{b_1b_2b_3}-\delta_c^{[a}\tilde{R}^{b_1b_2b_3]} \tag{5.4}$$

one can then show that the Jacobi identity between two 2-forms and momentum imposes [30]

$$[\tilde{R}^{ab},\tilde{R}^{cd}]=-2m\tilde{K}^{[a,b]cd}. \tag{5.5}$$

One can then show that the Jacobi identities involving the scalar operator $\tilde{R}$ require a non-trivial commutation relation between the deformed 7-form generator and the momentum operator, while the commutator between the 2-form and the 5-form generator has to be modified by a term proportional to the Og 1 $\tilde{K}^{a,b_1\dots b_6}$ the 6-form, which satisfies

$$[\tilde{K}^{a,b_1\dots b_6}, P_c] = \delta^a_c \tilde{R}^{b_1\dots b_6} - \delta^{[a}_c \tilde{R}^{b_1\dots b_6]}. \tag{5.6}$$

The result is

$$[\tilde{R}^{a_1\dots a_7}, P_b] = m\delta^{[a_1}_b \tilde{R}^{a_2\dots a_7]} \tag{5.7}$$

and

$$[\tilde{R}^{a_1a_2}, \tilde{R}^{b_1\dots b_5}] = \tilde{R}^{a_1a_2b_1\dots b_5} + m\tilde{K}^{[a_1,a_2]b_1\dots b_5}. \tag{5.8}$$

Finally, the Jacobi identities also impose that the commutator between the 9-form and momentum, as well as the commutator between the 2-form and the 7-form, must be modified. The result is

$$[\tilde{R}^{a_1\dots a_9}, P_b] = -5m\delta^{[a_1}_b \tilde{R}^{a_2\dots a_9]} \tag{5.9}$$

and

$$[\tilde{R}^{a_1a_2}, \tilde{R}^{b_1\dots b_7}] = \tilde{R}^{a_1a_2b_1\dots b_7} - \frac{17}{7}m\tilde{K}^{[a_1,a_2]b_1\dots b_7}, \tag{5.10}$$

where $\tilde{K}^{a,b_1\dots b_8}$ is the Og 1 operator for the 8-form, satisfying

$$[\tilde{K}^{a,b_1\dots b_8}, P_c] = \delta^a_c \tilde{R}^{b_1\dots b_8} - \delta^{[a}_c \tilde{R}^{b_1\dots b_8]}. \tag{5.11}$$

All the other commutators are not modified, and they are as in eq. (5.1) with all operators replaced by deformed operators.

To summarise, we have shown that starting from the E_{11} algebra of eq. (5.1) and introducing the deformed 2-form generator which satisfies the commutation relation of eq. (5.2), the Jacobi identities determine completely the rest of the algebra. In particular, once the algebra is expressed in terms of the tilde generators, the only commutators that are modified with respect to eq. (5.1) are those of eqs. (5.5), (5.8) and (5.10), while the additional non-trivial commutation relations with P_a are given is eqs. (5.3), (5.7) and (5.9).

We now consider the group element

$$g = e^{x\cdot P} e^{\Phi_{\rm Og}\tilde{K}^{\rm Og}} e^{A_{a_1\dots a_9}\tilde{R}^{a_1\dots a_9}} \dots e^{A_{a_1\dots a_5}\tilde{R}^{a_1\dots a_5}} e^{A_{a_1\dots a_3}\tilde{R}^{a_1\dots a_3}} e^{A_{a_1a_2}\tilde{R}^{a_1a_2}} e^{A_a\tilde{R}^a} e^{\phi\tilde{R}} \tag{5.12}$$

where we denote with $\Phi_{\rm Og}$ the whole set of Og 1 field of the ten-dimensional massive IIA theory. Similarly we denote with $\tilde{K}^{\rm Og}$ the whole set of deformed ten-dimensional Og 1 operators, which we treat as commuting because we are ignoring the contribution of Og 2 generators for simplicity. One can compute the Maurer-Cartan form that results from the group element in eq. (5.12). The result is

$$\begin{aligned} g^{-1}\partial_\mu g = {} & e^{2\phi}P_\mu + \partial_\mu\phi\tilde{R} + (\partial_\mu A_a + mA_{\mu a} + \cdots)e^{\phi}\tilde{R}^a + (\partial_\mu A_{a_1a_2} + \cdots)e^{-\phi}\tilde{R}^{a_1a_2} \\ & + \left(\partial_\mu A_{a_1a_2a_3} - \partial_\mu A_{a_1a_2}A_{a_3} + \frac{m}{2}A_{\mu a_1}A_{a_2a_3} + \cdots\right)\tilde{R}^{a_1\dots a_3} \\ & + \left(\partial_\mu A_{a_1\dots a_5} + 2\partial_\mu A_{a_1a_2a_3}A_{a_4a_5} + \frac{m}{3}A_{\mu a_1}A_{a_2a_3}A_{a_4a_5} + \cdots\right)e^{-\phi}\tilde{R}^{a_1\dots a_5} \end{aligned}$$

$$
\begin{aligned}
&+\Big(\partial_\mu A_{a_1\ldots a_6} - \partial_\mu A_{a_1\ldots a_5}A_{a_6} - 2\partial_\mu A_{a_1a_2a_3}A_{a_4a_5}A_{a_6} + \partial_\mu A_{a_1a_2a_3}A_{a_4a_5a_6}\\
&-\frac{m}{3}A_{\mu a_1}A_{a_2a_3}A_{a_4a_5}A_{a_6} - mA_{\mu a_1\ldots a_6} + \cdots\Big)\tilde{R}^{a_1\ldots a_6}\\
&+\Big(\partial_\mu A_{a_1\ldots a_7} - \partial_\mu A_{a_1\ldots a_5}A_{a_6a_7} - \partial_\mu A_{a_1a_2a_3}A_{a_4a_5}A_{a_6a_7} - \frac{m}{12}A_{\mu a_1}A_{a_2a_3}A_{a_4a_5}A_{a_6a_7}\\
&+\ldots\Big)e^{-2\phi}\tilde{R}^{a_1\ldots a_7} + \Big(\partial_\mu A_{a_1\ldots a_8} + 3\partial_\mu A_{a_1\ldots a_7}A_{a_8} + 2\partial_\mu A_{a_1\ldots a_6}A_{a_7a_8}\\
&+\partial_\mu A_{a_1\ldots a_5}A_{a_6\ldots a_8} - 3\partial_\mu A_{a_1\ldots a_5}A_{a_6a_7}A_{a_8} - 3\partial_\mu A_{a_1a_2a_3}A_{a_4a_5}A_{a_6a_7}A_{a_8}\\
&+2\partial_\mu A_{a_1..a_3}A_{a_4\ldots a_6}A_{a_7a_8} - \frac{m}{4}A_{\mu a_1}A_{a_2a_3}A_{a_4a_5}A_{a_6a_7}A_{a_8} - 2mA_{\mu a_1\ldots a_6}A_{a_7a_8}\\
&+5mA_{\mu a_1\ldots a_8} + \cdots\Big)e^{-\phi}\tilde{R}^{a_1\ldots a_8} + \Big(\partial_\mu A_{a_1\ldots a_9} - \partial_\mu A_{a_1\ldots a_7}A_{a_8a_9}\\
&+\frac{1}{2}\partial_\mu A_{a_1\ldots a_5}A_{a_6a_7}A_{a_8a_9} + \frac{1}{3}\partial_\mu A_{a_1a_2a_3}A_{a_4a_5}A_{a_6a_7}A_{a_8a_9}\\
&+\frac{m}{60}A_{\mu a_1}A_{a_2a_3}A_{a_4a_5}A_{a_6a_7}A_{a_8a_9} + \cdots\Big)e^{-3\phi}\tilde{R}^{a_1\ldots a_9} + \cdots\,. \qquad (5.13)
\end{aligned}
$$

The dots at the end denote terms proportional to the higher level deformed E_{11} generators as well as all the Og generators, while the dots in each bracket denote the contributions from the Og 1 fields, which we did not write down explicitly because their contribution vanishes after antisymmetrisation of the μ index with the other indices. Indeed, the Og 1 fields in the group element of eq. (5.12) are $\Phi_{a,b_1\ldots b_n}$, for $n = 1, 2, 3, 5, \ldots$, satisfying $\Phi_{[a,b_1\ldots b_n]} = 0$. The inverse Higgs mechanism relates these Og fields to the deformed E_{11} fields in such a way that only the completely antisymmetric terms in (5.13) survive. These terms are

$$
\begin{aligned}
F_{a_1a_2} &= \partial_{[a_1}A_{a_2]} + mA_{a_1a_2}\\
F_{a_1a_2a_3} &= \partial_{[a_1}A_{a_2a_3]}\\
F_{a_1\ldots a_4} &= \partial_{[a_1}A_{a_2a_3a_4]} - \partial_{[a_1}A_{a_2a_3}A_{a_4]} + \frac{m}{2}A_{[a_1a_2}A_{a_3a_4]}\\
F_{a_1\ldots a_6} &= \partial_{[a_1}A_{a_2\ldots a_6]} + 2\partial_{[a_1}A_{a_2a_3a_4}A_{a_5a_6]} + \frac{m}{3}A_{[a_1a_2}A_{a_3a_4}A_{a_5a_6]}\\
F_{a_1\ldots a_7} &= \partial_{[a_1}A_{a_2\ldots a_7]} - \partial_{[a_1}A_{a_2\ldots a_6}A_{a_7]} - 2\partial_{[a_1}A_{a_2a_3a_4}A_{a_5a_6}A_{a_7]} + \partial_{[a_1}A_{a_2a_3a_4}A_{a_5a_6a_7]}\\
&\quad -\frac{m}{3}A_{[a_1a_2}A_{a_3a_4}A_{a_5a_6}A_{a_7]} - mA_{a_1\ldots a_7}\\
F_{a_1\ldots a_8} &= \partial_{[a_1}A_{a_2\ldots a_8]} - \partial_{[a_1}A_{a_2\ldots a_6}A_{a_7a_8]} - \partial_{[a_1}A_{a_2a_3a_4}A_{a_5a_6}A_{a_7a_8]}\\
&\quad -\frac{m}{12}A_{[a_1a_2}A_{a_3a_4}A_{a_5a_6}A_{a_7a_8]}\\
F_{a_1\ldots a_9} &= \partial_{[a_1}A_{a_2\ldots a_9]} + 3\partial_{[a_1}A_{a_2\ldots a_8}A_{a_9]} + 2\partial_{[a_1}A_{a_2\ldots a_7}A_{a_8a_9]} + \partial_{[a_1}A_{a_2\ldots a_6}A_{a_7\ldots a_9]}\\
&\quad -3\partial_{[a_1}A_{a_2\ldots a_6}A_{a_7a_8}A_{a_9]} - 3\partial_{[a_1}A_{a_2a_3a_4}A_{a_5a_6}A_{a_7a_8}A_{a_9]} + 2\partial_{[a_1}A_{a_2..a_4}A_{a_5\ldots a_7}A_{a_8a_9]}\\
&\quad -\frac{m}{4}A_{[a_1a_2}A_{a_3a_4}A_{a_5a_6}A_{a_7a_8}A_{a_9]} - 2mA_{[a_1\ldots a_7}A_{a_8a_9]} + 5mA_{a_1\ldots a_9}\\
F_{a_1\ldots a_{10}} &= \partial_{[a_1}A_{a_2\ldots a_{10}]} - \partial_{[a_1}A_{a_2\ldots a_8}A_{a_9a_{10}]} + \frac{1}{2}\partial_{[a_1}A_{a_2\ldots a_6}A_{a_7a_8}A_{a_9a_{10}]}\\
&\quad +\frac{1}{3}\partial_{[a_1}A_{a_2a_3a_4}A_{a_5a_6}A_{a_7a_8}A_{a_9a_{10}]} + \frac{m}{60}A_{[a_1a_2}A_{a_3a_4}A_{a_5a_6}A_{a_7a_8}A_{a_9a_{10}]}. \qquad (5.14)
\end{aligned}
$$

These are the field-strengths of the fields of the massive IIA theory. Out of these field-strengths one can construct the field equations, which are duality relations between the various field-strengths. In particular, the 2-form $F_{a_1a_2}$ is dual to the 8-form $F_{a_1\dots a_8}$, the 3-form $F_{a_1a_2a_3}$ is dual to the 7-form $F_{a_1\dots a_7}$ and the 4-form $F_{a_1\dots a_4}$ is dual to the 6-form $F_{a_1\dots a_6}$, while the 9-form $F_{a_1\dots a_9}$ is dual to the derivative of the scalar and the 10-form $F_{a_1\dots a_{10}}$ is dual to the mass parameter m. All these relations are covariant under the local subalgebra of the non-linear realisation, which is SO(9, 1).

The gauge transformations of the fields arise in the non-linear realisation as rigid transformations of the group element, $g \to g_0 g$, as long as one includes the Og generators. One obtains

$$\begin{aligned}
\delta A_a &= \partial_a \Lambda - m\Lambda_a \\
\delta A_{a_1a_2} &= \partial_{[a_1}\Lambda_{a_2]} \\
\delta A_{a_1a_2a_3} &= \partial_{[a_1}\Lambda_{a_2a_3]} + \partial_{[a_1}\Lambda A_{a_2a_3]} - m\Lambda_{[a_1}A_{a_2a_3]} \\
\delta A_{a_1\dots a_5} &= \partial_{[a_1}\Lambda_{a_2\dots a_5]} - \partial_{[a_1}\Lambda A_{a_2a_3}A_{a_4a_5]} - 2\partial_{[a_1}\Lambda_{a_2}A_{a_3a_4a_5]} + m\Lambda_{[a_1}A_{a_2a_3}A_{a_4a_5]} \\
\delta A_{a_1\dots a_6} &= \partial_{[a_1}\Lambda_{a_2\dots a_6]} + \partial_{[a_1}\Lambda_{a_2a_3}A_{a_4a_5a_6]} - \partial_{[a_1}\Lambda A_{a_2\dots a_6]} - \partial_{[a_1}\Lambda A_{a_2a_3}A_{a_4a_5a_6]} \\
&\quad + m\Lambda_{a_1\dots a_6} + m\Lambda_{[a_1}A_{a_2\dots a_6]} + m\Lambda_{[a_1}A_{a_2a_3}A_{a_4a_5a_6]} \\
\delta A_{a_1\dots a_7} &= \partial_{[a_1}\Lambda_{a_2\dots a_7]} - \frac{1}{3}\partial_{[a_1}\Lambda A_{a_2a_3}A_{a_4a_5}A_{a_6a_7]} + \partial_{[a_1}\Lambda_{a_2}A_{a_3\dots a_7]} \\
&\quad + \frac{1}{3}m\Lambda_{[a_1}A_{a_2a_3}A_{a_4a_5}A_{a_6a_7]} \\
\delta A_{a_1\dots a_8} &= \partial_{[a_1}\Lambda_{a_2\dots a_8]} + \partial_{[a_1}\Lambda_{a_2a_3}A_{a_4\dots a_8]} + 3\partial_{[a_1}\Lambda A_{a_2\dots a_8]} + \partial_{[a_1}\Lambda A_{a_2a_3a_4}A_{a_5a_6}A_{a_7a_8]} \\
&\quad - 2\partial_{[a_1}\Lambda_{a_2}A_{a_3\dots a_8]} - 5m\Lambda_{a_1\dots a_8} - 3m\Lambda_{[a_1}A_{a_2\dots a_8]} - m\Lambda_{[a_1}A_{a_2a_3a_4}A_{a_5a_6}A_{a_7a_8]} \\
\delta A_{a_1\dots a_9} &= \partial_{[a_1}\Lambda_{a_2\dots a_9]} - \frac{1}{12}\partial_{[a_1}\Lambda A_{a_2a_3}A_{a_4a_5}A_{a_6a_7}A_{a_8a_9]} + \partial_{[a_1}\Lambda_{a_2}A_{a_3\dots a_9]} \\
&\quad + \frac{1}{12}m\Lambda_{[a_1}A_{a_2a_3}A_{a_4a_5}A_{a_6a_7}A_{a_8a_9]}.
\end{aligned} \tag{5.15}$$

In [33] the supersymmetry transformations of all the forms and dual forms of the massive IIA theory where determined. The supersymmetry algebra closes on all the local symmetries of the theory, and this was used to determine all the gauge transformations and the field-strengths of the various forms, as well as their duality relations. These forms are exactly those predicted by E_{11}. One can show that the field strengths and gauge transformations of [33] coincide with those given in eqs. (5.14) and (5.15) up to field redefinitions. The fact that using simple algebraic techniques one can easily determine these quantities proves the power of the E_{11} formulation of maximal supergravities and of the methods explained in this paper. In the next section we will apply these methods to the case of maximal gauged supergravity in five dimensions, deriving again the results of [20].

6 E_{11} and gauged five-dimensional supergravity

In section 4 we derived the algebra $\tilde{E}_{11,9}^{\text{local}}$ associated to the Scherk-Schwarz dimensional reduction of the IIB theory to nine dimensions. After deriving the field-strengths of the theory from a ten-dimensional group element with a given dependence on the 10th coordinate

y, we have shown that the same results can be obtained directly in nine dimensions. Indeed from the nine-dimensional perspective the fact that the ten-dimensional group element has a non-trivial y dependence translates in having generators of the nine-dimensional theory that are deformed with respect to the massless case. We have shown that the algebra of these deformed generators is uniquely fixed by the Jacobi identities, and in deriving the deformed algebra in this way one never makes use of the fact that the theory has a ten-dimensional origin. This approach was indeed taken in the previous section, where we derived the algebra $\tilde{E}^{\rm local}_{11,10A}$ corresponding to the massive IIA theory by requiring the closure of the Jacobi identities. From this algebra we have then derived the field-strengths of all the forms in the theory.

In this section we will perform precisely the same analysis for the case of maximal gauged supergravity in five dimensions. We will derive the algebra $\tilde{E}^{\rm local}_{11,5}$ and from it we will determine the field-strengths of the forms in the theory. The analysis follows exactly the same steps as we have shown in the previous section for the case of the massive IIA theory in ten dimensions. We will first review the E_{11} algebra as decomposed with respect to its $GL(5,\mathbb{R})\otimes E_6$ subalgebra [14] which is relevant for the five-dimensional analysis. This corresponds to deleting node 5 in the Dynkin diagram of figure 1. We will then consider the algebra of the deformed generators which occur in the description of the gauged theory from the E_{11} perspective. The commutation relations of these generators are completely fixed by imposing Jacobi identities. The resulting algebra is such that the non-linear realisation determines completely all the field-strengths of gauged maximal supergravity in five dimensions. A different approach to gauge supergravities, based on E_{10}, was presented in [34] for the three-dimensional case.

We now review the E_{11} commutation relations of the form generators up to the 4-form included that occur in the decomposition of E_{11} with respect to $GL(5,\mathbb{R})\otimes E_6$ [14]. These generators are

$$R^\alpha \qquad R^{a,M} \qquad R^{ab}{}_M \qquad R^{abc,\alpha} \qquad R^{abcd}{}_{[MN]}, \tag{6.1}$$

where R^α, $\alpha = 1,\ldots,78$ are the E_6 generators, and an upstairs M index, $M = 1,\ldots,27$, corresponds to the $\overline{\mathbf{27}}$ representation of E_6, a downstairs M index to the $\mathbf{27}$ of E_6 and a pair of antisymmetric downstairs indices $[MN]$ correspond to the $\overline{\mathbf{351}}$. The commutation relations for the E_6 generators is

$$[R^\alpha, R^\beta] = f^{\alpha\beta}{}_\gamma R^\gamma, \tag{6.2}$$

where $f^{\alpha\beta}{}_\gamma$ are the structure constants of E_6. The commutation relations of R^α with all the other generators is determined by the E_6 representations that they carry. This gives

$$\begin{aligned}
[R^\alpha, R^{a,M}] &= (D^\alpha)_N{}^M R^{a,N} \\
[R^\alpha, R^{ab}{}_M] &= -(D^\alpha)_M{}^N R^{ab}{}_N \\
[R^\alpha, R^{abc,\beta}] &= f^{\alpha\beta}{}_\gamma R^{abc,\gamma} \\
[R^\alpha, R^{abcd}{}_{[MN]}] &= -(D^\alpha)_M{}^P R^{abcd}{}_{[PN]} - (D^\alpha)_N{}^P R^{abcd}{}_{[MP]},
\end{aligned} \tag{6.3}$$

where $(D^\alpha)_N{}^M$ obey

$$[D^\alpha, D^\beta]_M{}^N = f^{\alpha\beta}{}_\gamma (D^\gamma)_M{}^N \quad . \tag{6.4}$$

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The commutation relations of all the other generators are

$$\begin{aligned}
[R^{a,M}, R^{b,N}] &= d^{MNP} R^{ab}{}_P \\
[R^{a,N}, R^{bc}{}_M] &= g_{\alpha\beta} (D^\alpha)_M{}^N R^{abc,\beta} \\
[R^{ab}{}_M, R^{cd}{}_N] &= R^{abcd}{}_{[MN]} \\
[R^{a,P}, R^{bcd,\alpha}] &= S^{\alpha P[MN]} R^{abcd}{}_{[MN]},
\end{aligned} \tag{6.5}$$

where d^{MNP} is the symmetric invariant tensor of E_6 and $g_{\alpha\beta}$ is the Cartan-Killing metric of E_6. $S^{\alpha P[MN]}$ is also an invariant tensor, which the Jacobi identities fix to be

$$S^{\alpha P[MN]} = -\frac{1}{2} D_Q^{\alpha}{}^{[M} d^{N]QP}, \tag{6.6}$$

and which satisfies the further identity

$$g_{\alpha\beta} D_Q^{\alpha}{}^{(P} S^{\beta R)[MN]} = -\frac{1}{2} \delta_Q^{[M} d^{N]PR}. \tag{6.7}$$

One can show that all the Jacobi identities involving the generators in eq. (6.1) are satisfied using the commutators listed above.

To obtain the field-strengths of the massless theory, one introduces the Og generators, that encode the gauge transformations of all the fields. We focus in particular on the Og 1 generators for the E_{11} generators listed in eq. (6.1), that are

$$K^{a,b,M} \qquad K^{a,b_1 b_2}{}_M \qquad K^{a,b_1 b_2 b_3,\alpha} \qquad K^{a,b_1\dots b_4}{}_{[MN]}, \tag{6.8}$$

and whose commutators with the momentum operator are

$$\begin{aligned}
[K^{a,b,M}, P_c] &= \delta_c^{(a} R^{b),M} \\
[K^{a,b_1 b_2}{}_M, P_c] &= \delta_c^a R^{b_1 b_2}{}_M - \delta_c^{[a} R^{b_1 b_2]}{}_M \\
[K^{a,b_1 b_2 b_3,\alpha}, P_c] &= \delta_c^a R^{b_1 b_2 b_3,\alpha} - \delta_c^{[a} R^{b_1 b_2 b_3],\alpha} \\
[K^{a,b_1\dots b_4}{}_{[MN]}, P_c] &= \delta_c^a R^{b_1\dots b_4}{}_{[MN]} - \delta_c^{[a} R^{b_1\dots b_4]}{}_{[MN]}.
\end{aligned} \tag{6.9}$$

We then write down the group element

$$g = e^{x\cdot P} e^{\Phi_{\rm Og} K^{\rm Og}} e^{A_{a_1\dots a_4}^{[MN]} R^{a_1\dots a_4}_{[MN]}} e^{A_{a_1 a_2 a_3,\alpha} R^{a_1 a_2 a_3,\alpha}} e^{A^M_{a_1 a_2} R^{a_1 a_2}_M} e^{A_{a,M} R^{a,M}} e^{\phi_\alpha R^\alpha}, \tag{6.10}$$

where we denote with $K^{\rm Og}$ all the Og 1 generators listed in eq. (6.8) and with $\Phi_{\rm Og}$ their corresponding fields. One can then compute the Maurer-Cartan form, and use the inverse Higgs mechanism to fix all the Og 1 fields in terms of derivatives of the E_{11} fields, in such a way that only the completely antisymmetric terms in the Maurer-Cartan form survive. These quantities are the gauge-invariant field-strengths of the massless theory obtained in [20], which we list here

$$\begin{aligned}
F_{a_1 a_2,M} &= \partial_{[a_1} A_{a_2],M} \\
F^M_{a_1 a_2 a_3} &= \partial_{[a_1} A^M_{a_2 a_3]} + \frac{1}{2} \partial_{[a_1} A_{a_2,N} A_{a_3],P} d^{MNP}
\end{aligned}$$

$$
\begin{aligned}
F^{\alpha}_{a_1a_2a_3a_4} &= \partial_{[a_1}A^{\alpha}_{a_2a_3a_4]} - \frac{1}{6}\partial_{[a_1}A_{a_2,M}A_{a_3,N}A_{a_4],P}d^{MNQ}D_Q^{\alpha\,P} - \partial_{[a_1}A^M_{a_2a_3}A_{a_4],N}D_M^{\alpha\,N} \\
F^{MN}_{a_1\dots a_5} &= \partial_{[a_1}A^{MN}_{a_2\dots a_5]} - \frac{1}{24}\partial_{[a_1}A_{a_2,P}A_{a_3,Q}A_{a_4,R}A_{a_5],S}d^{PQT}D_T^{\alpha\,R}S^{\alpha S[MN]} \\
&\quad -\frac{1}{2}\partial_{[a_1}A^P_{a_2a_3}A_{a_4,Q}A_{a_5],R}D_P^{\alpha\,Q}S^{\alpha R[MN]} + \frac{1}{2}\partial_{[a_1}A^{[M}_{a_2a_3}A^{N]}_{a_4a_5]} \\
&\quad +\partial_{[a_1}A^{\alpha}_{a_2a_3a_4}A_{a_5],P}S^{\alpha P[MN]}
\end{aligned}
\tag{6.11}
$$

for completeness.

We now consider the deformed case. We take as our set of generators that of eqs. (6.1) and (6.8), but to indicate that these generators themselves have been deformed, we denote them with a tilde. The commutation relations receive order g corrections with respect to the massless ones, where g is the deformation parameter. We start from the commutation relation between the deformed vector generator and the momentum operator. We impose this to be

$$
[\tilde{R}^{a,M}, P_b] = -g\delta^a_b\Theta^M_\alpha R^\alpha. \tag{6.12}
$$

The quantity Θ^M_α turns out to be the embedding tensor [7], and the generators $\Theta^M_\alpha R^\alpha$ are the generators of the subgroup G of E_6 that is gauged. These generators belong to the algebra $\tilde{E}^{\text{local}}_{11,5}$. We now show that all the commutation relations of the deformed operators among themselves and with the momentum operators are uniquely fixed by Jacobi identities. From the resulting algebra we construct the non-linear realisation whose Maurer-Cartan form gives the field-strengths of the gauged theory.

We first consider the Jacobi identity between $\Theta^M_\alpha R^\alpha$, $\tilde{R}^{a,N}$ and P_b. Defining

$$
X_P^{MN} = \Theta^M_\alpha D_P^{\alpha\,N} \tag{6.13}
$$

one gets

$$
\Theta^M_\alpha \Theta^N_\beta f^{\alpha\beta}{}_\gamma - \Theta^P_\gamma X_P^{MN} = 0, \tag{6.14}
$$

which turns out to be the condition that the embedding tensor is invariant under the gauge group. We then write the commutator of the 2-form $\tilde{R}^{ab}{}_M$ with P_c as

$$
[\tilde{R}^{ab}{}_M, P_c] = -2gW_{MN}\delta_c^{[a}\tilde{R}^{b],N}, \tag{6.15}
$$

which defines the antisymmetric tensor W_{MN}. The Jacobi identity involving $\tilde{R}^{ab}{}_M$, P_c and P_d gives

$$
W_{MN}\Theta^N_\alpha = 0. \tag{6.16}
$$

The Jacobi identity involving the operators $\tilde{R}^{a,M}$, $\tilde{R}^{b,N}$ and P_c gives

$$
[\tilde{R}^{a,M}, \tilde{R}^{b,N}] = d^{MNP}\tilde{R}^{ab}{}_P - 2gX_P^{[MN]}\tilde{K}^{a,b,P}, \tag{6.17}
$$

using the fact that the Og 1 for the vector $\tilde{K}^{a,b,M}$ is symmetric in ab and satisfies

$$
[\tilde{K}^{a,b,M}, P_c] = \delta_c^{(a}\tilde{R}^{b),M}. \tag{6.18}
$$

To get eq. (6.17) one has also to impose

$$
X_P^{(MN)} = -W_{PQ}d^{QMN}. \tag{6.19}
$$

The Jacobi identity between $\tilde{R}^{ab}{}_M$, $\Theta_\alpha^N R^\alpha$ and P_c gives

$$X_{[P}^{MN} W_{Q]N} = 0, \tag{6.20}$$

which is the condition that the tensor W_{MN} is invariant under the gauge subgroup G. The Jacobi identity involving $\tilde{R}^{a,M}$, $\tilde{R}^{bc}{}_N$ and P_d gives

$$[\tilde{R}^{a,M}, \tilde{R}^{bc}{}_N] = (D^\alpha)_N{}^M \tilde{R}^{abc}{}_\alpha + \frac{3}{2} g \left(X_N^{MP} + \frac{1}{3} X_N^{PM} \right) \tilde{K}^{a,bc}{}_P \tag{6.21}$$

and

$$[\tilde{R}^{abc}{}_\alpha, P_d] = -g \Theta_\alpha^M \delta_d^{[a} \tilde{R}^{bc]}{}_M, \tag{6.22}$$

where the Og 1 operator $\tilde{K}^{a,bc}{}_M$ satisfies

$$[\tilde{K}^{a,bc}{}_M, P_d] = \delta_d^a \tilde{R}^{bc}{}_M - \delta_d^{[a} \tilde{R}^{bc]}{}_M - \frac{2}{3} g W_{MN} (\delta_d^b \tilde{K}^{a,c,N} - \delta_d^c \tilde{K}^{a,b,N}). \tag{6.23}$$

Proceeding this way, one can determine all the commutators requiring the closure of the Jacobi identities. This gives

$$[\tilde{R}^{ab}{}_M, \tilde{R}^{cd}{}_N] = \tilde{R}^{abcd}{}_{MN} - 4g W_{(M|P|} D^\alpha_{N)}{}^P \tilde{K}^{[a,b]cd}{}_\alpha, \tag{6.24}$$

$$[\tilde{R}^{a,M}, \tilde{R}^{bcd}{}_\alpha] = S_\alpha^{M[NP]} \tilde{R}^{abcd}{}_{MN} - g \left(f^{\beta\gamma}{}_\alpha \Theta_\gamma^M + \frac{1}{3} D_P^{\beta\, M} \Theta_\alpha^P \right) \tilde{K}^{a,bcd}{}_\beta \tag{6.25}$$

and

$$[\tilde{R}^{a_1 \dots a_4}{}_{MN}, P_b] = -4g W_{[M|P|} D^\alpha_{N]}{}^P \delta_b^{[a_1} \tilde{R}^{a_2 a_3 a_4]}{}_\alpha, \tag{6.26}$$

where we introduce the Og 1 generator for the 3-form $\tilde{K}^{a,b_1 b_2 b_3}{}_\alpha$, satisfying

$$[\tilde{K}^{a,b_1 b_2 b_3}{}_\alpha, P_c] = \delta_c^a \tilde{R}^{b_1 b_2 b_3}{}_\alpha - \delta_c^{[a} \tilde{R}^{b_1 b_2 b_3]}{}_\alpha - \frac{3}{4} g \Theta_\alpha^M \delta_c^{[b_1} \tilde{K}^{|a|, b_2 b_3]}. \tag{6.27}$$

In order to get these results one has to impose an additional constraint

$$f^{\alpha\beta}{}_\gamma \Theta_\beta^Q - D_P^{\alpha\, Q} \Theta_\gamma^P = 4 D_M^\alpha{}^P W_{PN} g_{\beta\gamma} S^{\beta Q[MN]}, \tag{6.28}$$

which shows that the embedding tensor and W_{MN} are related by the invariant tensor $S^{\alpha P[MN]}$, and thus belong to the same representation of E_6, which is the $\overline{\mathbf{351}}$.

To summarise, we have determined the commutation relations satisfied by the deformed E_{11} p-form generators, the corresponding deformed Og 1 generators and the momentum operator of the five-dimensional massive theory starting from eq. (6.12) and imposing the closure of the Jacobi identities. We will now determine the field-strengths of the 1-forms, 2-forms and 3-forms of the theory using these results. We consider the group element

$$g = e^{x \cdot P} e^{\Phi_{\rm Og} \tilde{K}^{\rm Og}} e^{A^{[MN]}_{a_1 \dots a_4} \tilde{R}^{a_1 \dots a_4}_{[MN]}} e^{A_{a_1 a_2 a_3, \alpha} \tilde{R}^{a_1 a_2 a_3, \alpha}} e^{A^M_{a_1 a_2} \tilde{R}^{a_1 a_2}_M} e^{A_{a,M} \tilde{R}^{a,M}} e^{\phi_\alpha R^\alpha}, \tag{6.29}$$

where as in the massless case we denote with $\tilde{K}^{\rm Og}$ all the deformed Og 1 generators and with $\Phi_{\rm Og}$ the corresponding fields. One can then compute the Maurer-Cartan form, and use the inverse Higgs mechanism to fix all the Og 1 fields in such a way that only the completely antisymmetric terms in the Maurer-Cartan form survive. To compute the field-strengths, it is thus sufficient to consider only the $\tilde{R}$ operators and P_a in the group element above. The final result is

$$\begin{aligned}
F_{a_1a_2,M} &= \partial_{[a_1}A_{a_2],M} + \frac{1}{2}gX_M^{[NP]}A_{[a_1,N}A_{a_2],P} - 2gW_{MN}A^N_{a_1a_2} \\
F^M_{a_1a_2a_3} &= \partial_{[a_1}A^M_{a_2a_3]} + \frac{1}{2}\partial_{[a_1}A_{a_2,N}A_{a_3],P}d^{MNP} - 2gX_P^{(MN)}A^P_{[a_1a_2}A_{a_3],N} \\
&\quad + \frac{1}{6}gX_R^{[NP]}d^{RQM}A_{[a_1,N}A_{a_2,P}A_{[a_3],Q} + g\Theta_\alpha^M A^\alpha_{a_1a_2a_3} \\
F^\alpha_{a_1\ldots a_4} &= \partial_{[a_1}A^\alpha_{a_2\ldots a_4]} - \frac{1}{6}\partial_{[a_1}A_{a_2,M}A_{a_3,N}A_{a_4],P}d^{MNQ}D^{\alpha}_Q{}^P - \partial_{[a_1}A^M_{a_2a_3}A_{a_4],N}D^\alpha_M{}^N \\
&\quad + gD^\alpha_M{}^P\Theta^M_\beta A_{[a_1,P}A^\beta_{a_2\ldots a_4]} + 4gD^\alpha_M{}^PW_{PN}A^{MN}_{a_1\ldots a_4} - gD^\alpha_M{}^PW_{PN}A^M_{[a_1a_2}A^N_{a_3a_4]} \\
&\quad - gD^\alpha_M{}^PX_Q^{(MR)}A_{[a_1,P}A_{a_2,R}A^Q_{a_3a_4]} \\
&\quad - \frac{1}{24}gX_R^{[MN]}d^{RPS}D^{\alpha Q}_S A_{[a_1,M}A_{a_2,N}A_{a_3,P}A_{a_4],Q}
\end{aligned} \tag{6.30}$$

These are the field-strengths of the five-dimensional gauged maximal supergravity [20]. One can also derive the gauge transformations of the fields from the non-linear realisation as they arise as rigid transformations of the group element, $g \to g_0 g$, as long as one includes the Og generators. The result is

$$\begin{aligned}
\delta A_{a,N} &= \partial_a\Lambda_N - g\Lambda_S X_N^{SM}A_{aM} + 2gW_{MP}\Lambda^P_a \\
\delta A^N_{a_1a_2} &= \partial_{[a_1}\Lambda^N_{a_2]} + \frac{1}{2}\partial_{[a_1}\Lambda_S A_{a_2]T}d^{STN} + g\Lambda_S X_M^{SN}A^M_{a_1a_2} + 2W_{SP}\Lambda^P_{[a_1}A_{a_2]T}d^{STN} \\
&\quad - g\Lambda^\alpha_{a_1a_2}\Theta^N_\alpha \\
\delta A^\alpha_{a_1a_2a_3} &= \partial_{[a_1}\Lambda^\alpha_{a_2a_3]} + \partial_{[a_1}\Lambda_M A^N_{a_2a_3]}D^\alpha_N{}^M + \frac{1}{6}\partial_{[a_1}\Lambda_M A_{a_2,N}A_{a_3],P}d^{MNQ}D^\alpha_Q{}^P \\
&\quad - g\Lambda_P\Theta^P_\beta f^{\alpha\beta}{}_\gamma A^\gamma_{a_1a_2a_3} + 2gW_{MP}\Lambda^P_{[a_1}D^\alpha_N{}^M A^N_{a_2a_3]} \\
&\quad + \frac{1}{3}gW_{MR}\Lambda^R_{[a_1}d^{MNQ}D^\alpha_Q{}^P A_{a_2,N}A_{a_3]} \\
&\quad - 4gD^\alpha_M{}^PW_{PN}\Lambda^{MN}_{a_1a_2a_3}.
\end{aligned} \tag{6.31}$$

In [20] these transformations were derived both from E_{11} and from requiring the closure of the supersymmetry algebra. Indeed, the commutator of two supersymmetry transformations on these fields gives the gauge transformations above provided that the fields are related by dualities. In particular the 1-forms are dual to 2-forms while the 3-forms are dual to scalars in five dimensions. The field-strength of the 4-form is dual to the mass parameter. The field strengths and gauge transformations obtained here precisely agree with those obtained by supersymmetry.

In the above we have taken Jacobi identities with all the deformed generators of eqs. (6.1) and (6.8) with the exception of R^α, but we have instead restricted our use to $\Theta^M_\alpha R^\alpha$. Using the Jacobi identities for R^α would lead to results that are too strong. A solution to this problem, at least at low levels, requires adding to spacetime the scalar charges in the l multiplet, as was done in [20]. In this case the Jacobi identities are automatically satisfied. Adding the higher charges in the l multiplet may also resolve this problem at higher level.

To summarise, we have thus shown that the methods explained in this paper give an extremely fast way of computing the field strengths of all the forms and dual forms of five-dimensional gauged maximal supergravity. These methods can be easily generalised to any dimension, providing a remarkably efficient way of determining the gauge algebra of any massless or massive theory with maximal supersymmetry.

7 The dual graviton

Any very-extended Kac-Moody algebra, when decomposed in terms of a $GL(D,\mathbb{R})$ subalgebra which one associates to the non-linear realisation of gravity, contains a generator with $D-2$ indices in the hook Young tableaux irreducible representation with $D-3$ completely antisymmetric indices, that is $R^{a,b_1\ldots b_{D-3}}$ with $R^{[a,b_1\ldots b_{D-3}]}=0$ (in the case of E_{11} decomposed in terms of $GL(11,\mathbb{R})$, this generator is $R^{a,b_1\ldots b_8}$). The field associated to this generator in the non-linear realisation has the degrees of freedom of the dual graviton. The Kac-Moody algebra therefore describes together the graviton and the dual graviton. In this section we will consider the Og operators for the dual graviton. We will focus on the four-dimensional case, in which the dual graviton generator R^{ab} is symmetric in its two indices.

In subsection 7.1 we will first consider the case of the dual graviton in flat space. This corresponds to considering the dual graviton generator by itself, together with its corresponding Og generators. This does not arise from any very-extended Kac-Moody algebra. A field theory description of a linearised dual graviton is known to exist, and its field equations in four dimensions were first obtained by Curtright [35]. For subsequent developments see [1, 36].

We then consider the dual graviton coupled to gravity. The simplest very-extended Kac-Moody algebra whose non-linear realisation gives rise to a four-dimensional theory is the algebra A_1^{+++}, whose Dynkin diagram is shown in figure 5. The corresponding spectrum does not contain any form. We will show that there is no consistent solution of the inverse Higgs mechanism that leaves a propagating dual graviton. We will also consider the case of E_{11} in four dimensions, which corresponds to deleting node 4 in the diagram of figure 1, leading to the internal symmetry algebra E_7. In this case we will show that even considering linearised gravity, that is neglecting the $GL(4,\mathbb{R})$ generators and the corresponding Og generators, and only considering interactions of the dual graviton with matter, one is left with no consistent field strength for the dual graviton. This result is consistent with [37], where it was shown that it is impossible to write down a dual Riemann tensor in the presence of matter even when gravity is treated at the linearised level.

In the first subsection we will only consider the algebra of R^{ab} and all the corresponding Og generators. The Maurer-Cartan form, that can be thought as the Maurer-Cartan form of A_1^{+++} or E_{11} truncated to this sector, leads to invariant quantities that can be constrained by means of the inverse Higgs mechanism to generate the Riemann tensor for the linearised dual graviton. In the second subsection we will then consider the case of dual graviton

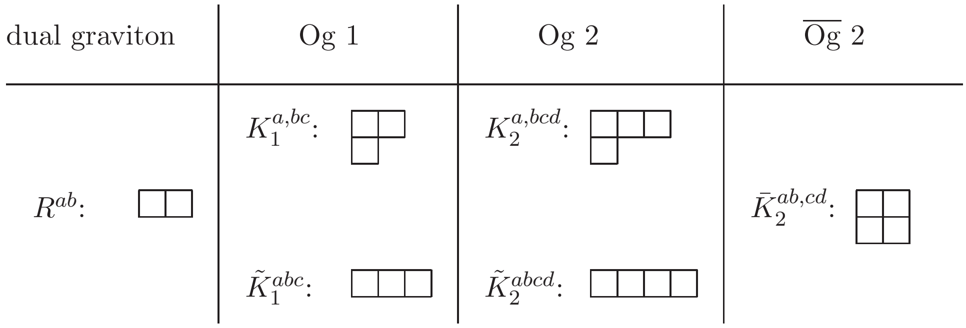

Figure 3. The Young tableaux of the first Og and $\overline{\text{Og}}$ generators for the dual graviton in four dimensions.

coupled to gravity, which corresponds to the algebra A_1^{+++}, and in the third subsection we will consider the E_{11} case of the dual graviton coupled to vectors with linearised gravity.

7.1 The dual graviton in four dimensions

In this subsection we want to consider the dual graviton alone, that is without introducing the generator associated to the graviton or any other matter generator. We want to show that one can introduce suitable Og generators for the dual graviton in such a way that gives rise to a consistent field strength and consistent gauge transformations. The dual graviton generator in four dimensions is a generator with 2 symmetric indices R^{ab}. Following the notation of the previous section, we define two Og 1 generators $K_1^{a,bc}$ and $\tilde{K}_1^{abc}$ in the irreducible $GL(4,\mathbb{R})$ representations defined as

$$\begin{aligned} K_1^{a,bc} &= K_1^{a,(bc)} \qquad\qquad K_1^{(a,bc)} = 0 \\ \tilde{K}_1^{abc} &= \tilde{K}_1^{(abc)}, \end{aligned} \tag{7.1}$$

and whose corresponding Young tableaux are shown in figure 3. Note that the sum of these two representations corresponds to an object with three indices, symmetric under the exchange of two of them and with no further constraint. These operators satisfy the commutation relations

$$\begin{aligned} [K_1^{a,bc}, P_d] &= \delta_d^a R^{bc} - \delta_d^{(a} R^{bc)} \\ [\tilde{K}_1^{abc}, P_d] &= \delta_d^{(a} R^{bc)}, \end{aligned} \tag{7.2}$$

while we take R^{ab} as commuting with P_a. We also take R^{ab} as commuting with itself because we are considering the dual graviton alone (for instance in A_1^{+++} the dual graviton is a generator at level 1, and therefore the commutator of two dual graviton generators leads to an operator at level 2). Also all the dual graviton Og generators are taken to commute with each other, and to commute with R^{ab} as well.

We now consider the Og 2 operators. These are $K_2^{a,bcd}$ and $\tilde{K}_2^{abcd}$ in the $GL(4,\mathbb{R})$ representations

$$
\begin{aligned}
K_2^{a,bcd} &= K_2^{a,(bcd)} \qquad K_2^{(a,bcd)} = 0 \\
\tilde{K}_2^{abcd} &= \tilde{K}_2^{(abcd)},
\end{aligned}
\tag{7.3}
$$

whose Young tableaux are shown in figure 3, and their commutation relation with P_a is

$$
\begin{aligned}
[K_2^{a,bcd}, P_e] &= \delta_e^{(b} K_1^{|a|,cd)} + \frac{2}{3}(\delta_e^a \tilde{K}_1^{bcd} - \delta_e^{(a} \tilde{K}_1^{bcd)}) \\
[\tilde{K}_2^{abcd}, P_e] &= \delta_e^{(a} \tilde{K}_1^{bcd)}.
\end{aligned}
\tag{7.4}
$$

The coefficient $\frac{2}{3}$ in the first commutator can be obtained from the Jacobi identity involving $K_2^{a,bcd}$ and two P's.

We will now compute the Maurer-Cartan form, and we will first consider only the contribution from dual graviton and the Og 1 fields, while the Og 2 fields will be included later. We thus consider the group element

$$
g = e^{x\cdot P} e^{\Phi^1_{a,bc} K_1^{a,bc}} e^{\tilde{\Phi}^1_{abc} \tilde{K}_1^{abc}} e^{A_{ab} R^{ab}},
\tag{7.5}
$$

from which one computes the Maurer-Cartan form

$$
g^{-1}\partial_\mu g = P_\mu + (\partial_\mu A_{ab} - \Phi^1_{\mu,ab} - \tilde{\Phi}^1_{\mu ab}) R^{ab} + \cdots ,
\tag{7.6}
$$

which is invariant under

$$
\begin{aligned}
\delta A_{ab} &= a_{ab} + x^c b_{c,ab} + x^c \tilde{b}_{abc} \\
\delta \Phi^1_{a,bc} &= b_{a,bc} \\
\delta \tilde{\Phi}^1_{a,bc} &= \tilde{b}_{a,bc}.
\end{aligned}
\tag{7.7}
$$

The first of eqs. (7.7) is reproducing the gauge transformation for the dual graviton in flat space,

$$
\delta A_{ab} = \partial_{(a} \Lambda_{b)}
\tag{7.8}
$$

at linear order in x, that is quadratic order in x for the gauge parameter Λ_a,

$$
\Lambda_a = a_{ab} x^b - b_{a,bc} x^b x^c + \frac{1}{2} \tilde{b}_{abc} x^b x^c.
\tag{7.9}
$$

One can solve for inverse Higgs in such a way that the whole Maurer-Cartan form proportional to R^{ab} vanishes compatibly with the symmetries. This corresponds to fixing

$$
\Phi^1_{\mu,ab} = \partial_\mu A_{ab} - \partial_{(\mu} A_{ab)}
\tag{7.10}
$$

and

$$
\tilde{\Phi}^1_{\mu,ab} = \partial_{(\mu} A_{ab)}.
\tag{7.11}
$$

The fact that reproducing the gauge transformations for A_{ab} at linear order in x allows one to eliminate completely the Maurer-Cartan form proportional to R^{ab} by means of the inverse Higgs mechanism corresponds to the fact that one cannot write a gauge invariant quantity at linear order in the derivatives. Note that there is a crucial difference here with respect to the non-linear realisation of gravity discussed in section 2. In that case the part of the Maurer-Cartan form proportional to the generators of the local subalgebra SO(D) gives the SO(D) connection, which becomes the spin connection once the inverse Higgs mechanism is applied. In this case the dual graviton field is already symmetric, and thus there is no corresponding local Lorentz symmetry. It is for this reason that at this level the Maurer-Cartan form vanishes once the inverse Higgs mechanism is applied.

We now consider the contribution from the Og 2 fields. We write the group element as

$$g = e^{x\cdot P} e^{\Phi^2_{a,bcd} K_2^{a,bcd}} e^{\tilde{\Phi}^2_{abcd} \tilde{K}_2^{abcd}} e^{\Phi^1_{a,bc} K_1^{a,bc}} e^{\tilde{\Phi}^1_{abc} \tilde{K}_1^{abc}} e^{A_{ab} R^{ab}}, \tag{7.12}$$

and obtain the corresponding Maurer-Cartan form

$$\begin{aligned} g^{-1}\partial_\mu g =\ & P_\mu + (\partial_\mu A_{ab} - \Phi^1_{\mu,ab} - \tilde{\Phi}^1_{\mu ab}) R^{ab} \\ & + (\partial_\mu \Phi^1_{a,bc} - \Phi^2_{a,\mu bc}) K_1^{a,bc} \\ & + \left(\partial_\mu \tilde{\Phi}^1_{abc} - \frac{2}{3}\Phi^2_{\mu,abc} - \tilde{\Phi}^2_{\mu abc} \right) \tilde{K}_1^{abc} + \cdots . \end{aligned} \tag{7.13}$$

Having introduced the Og 2 operators, the transformations of the field that leave the Maurer-Cartan form invariant acquire additional contributions, and in particular there is a term in the variation of A_{ab} which is quadratic in x. The result is

$$\begin{aligned} \delta A_{ab} &= a_{ab} + x^c b_{c,ab} + x^c \tilde{b}_{abc} + \frac{5}{6} x^c x^d c_{c,abd} - \frac{1}{2} x^c x^d c_{(c,ab)d} + \frac{1}{2} x^c x^d \tilde{c}_{abcd} \\ \delta \Phi^1_{a,bc} &= b_{a,bc} + x^d c_{a,bcd} - x^d c_{(a,bc)d} \\ \delta \tilde{\Phi}^1_{a,bc} &= \tilde{b}_{a,bc} + \frac{2}{3} x^d c_{d,abc} + x^d \tilde{c}_{abcd} \\ \delta \Phi^2_{a,bcd} &= c_{a,bcd} \\ \delta \tilde{\Phi}^2_{a,bcd} &= \tilde{c}_{a,bcd}. \end{aligned} \tag{7.14}$$

In particular the first of these variations is the most general gauge transformation for the field A_{ab} of the form (7.8) up to terms cubic in x.

We now apply the inverse Higgs mechanism, solving for the fields $\Phi^2_{a,bcd}$ and $\tilde{\Phi}^2_{abcd}$ in terms of A_{ab}. The result is

$$\begin{aligned} \Phi^2_{a,bcd} &= \frac{1}{4}[\partial_a\partial_b A_{cd} + \partial_a\partial_c A_{db} + \partial_a\partial_d A_{bc} - \partial_b\partial_c A_{ab} - \partial_b\partial_d A_{ac} - \partial_c\partial_d A_{ab}] \\ \tilde{\Phi}^2_{abcd} &= \partial_{(a}\partial_b A_{cd)} \quad . \end{aligned} \tag{7.15}$$

Plugging this into the Maurer-Cartan form, one notices that there is a non-vanishing term proportional to $K_1^{a,bc}$, that is

$$g^{-1}\partial_\mu g = P_\mu + \left(\frac{1}{3}\partial_\mu\partial_a A_{bc} - \frac{1}{3}\partial_\mu\partial_b A_{ac} - \frac{1}{3}\partial_a\partial_b A_{\mu c} + \frac{1}{3}\partial_b\partial_c A_{\mu a} \right) K_1^{a,bc} + \cdots . \tag{7.16}$$

This is indeed the Riemann tensor of linearised gravity $D_{ab,cd}$, which is a tensor in the window-like Young tableaux representation.

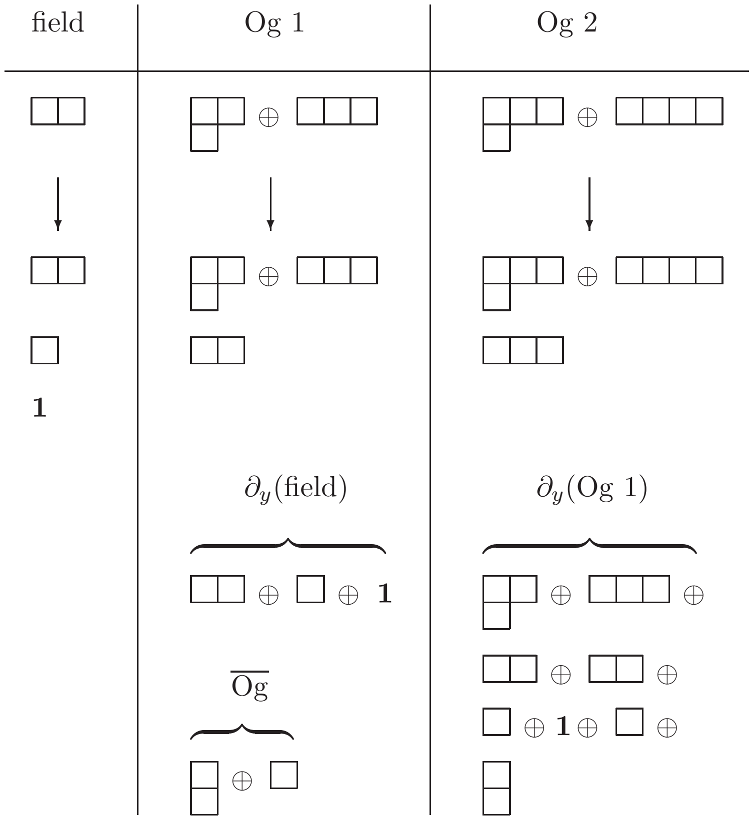

Figure 4. The dimensional reduction of the dual graviton and its Og fields. Each field is aligned horizontally with its corresponding Og fields. The rest of the Og n fields are associated to the y derivative of the Og $(n-1)$ fields. The dimensional reduction also produces $\overline{\text{Og}}$ 1 fields for the vector and the scalar.

One can introduce in the same way the higher Og generators, constructing in this way gauge invariant quantities which are derivatives of the Riemann tensor. The end result is thus

$$g^{-1}\partial_\mu g = P_\mu + D_{\mu a,bc} K_1^{a,bc} + \cdots \,, \tag{7.17}$$

where the dots correspond to derivatives of the dual graviton Riemann tensor

$$D_{ab,cd} = \frac{1}{3}\partial_a\partial_b A_{cd} - \frac{1}{6}\partial_a\partial_c A_{bd} - \frac{1}{6}\partial_a\partial_d A_{bc} - \frac{1}{6}\partial_b\partial_c A_{ad} - \frac{1}{6}\partial_b\partial_d A_{ac} + \frac{1}{3}\partial_c\partial_d A_{ab} \tag{7.18}$$

contracted with higher order Og generators. This shows that the linearised dual graviton admits a description in terms of R^{ab} and Og generators, and the corresponding Maurer-Cartan form contains the correct Riemann tensor, which can be used to construct the dynamics.

We now want to perform a dimensional reduction on a circle of coordinate y. We thus take the dual graviton and all the Og fields to be y independent. The representations of $GL(3,\mathbb{R})$ that arise in the three-dimensional compactified theory are shown in figure 4 for the dual graviton field and the first two Og fields. The circle dimensional reduction corresponds to the assumption that neither the dual graviton field nor the Og fields depend on y.

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The dual graviton A_{ab} in four dimensions has two symmetric indices, and after dimensional reduction it leads to an object with two symmetric indices A_{ab}, a vector $A_a = A_{ay}$ and a scalar $A = A_{yy}$. As the figure shows, the Og 1 fields can be divided in three sets. The first one contains the Og fields for the field with two symmetric indices A_{ab} and the vector A_a. This is precisely what we want in order to obtain the correct gauge transformations for the three-dimensional fields. The second set contains the same representations as the dimensionally reduced fields. The dy part of the Maurer-Cartan form contains these fields summed to the y derivative of the dimensionally reduced fields. Thus, from the requirement that the fields do not depend on y it follows that these Og fields can be put to zero using the inverse Higgs mechanism. Finally, the third set contains a field with two antisymmetric indices and a vector, and we call these fields the $\overline{\text{Og}}$ 1 fields for the vector and the scalars. These fields are the ones of interest to us in the following. The dimensional reduction of the Og 2 fields gives the Og 2 fields for the field with two symmetric indices and the vector, together with a set of fields in the same representations as the dimensionally reduced Og 1 fields, and again the dy part of the Maurer-Cartan form contains these Og 2 fields summed to the y derivative of all the Og 1 fields. Using y independence and the inverse Higgs mechanism one thus sets to zero these Og 2 fields.

We now explain the occurrence of the $\overline{\text{Og}}$ 1 fields and generators in the dimensional reduction. In the four-dimensional theory, once the inverse Higgs mechanism is applied the Maurer-Cartan form is given in eq. (7.17), and the first non-vanishing term is the dual graviton Riemann tensor, which is at second order in derivatives. Using the y-independence of the fields, the dimensional reduction of the Riemann tensor leads to the Riemann tensor for A_{ab} in three dimensions together with

$$\begin{aligned} D_{ab,cy} &= \frac{1}{2}\partial_a F_{bc} + \frac{1}{3}\partial_{(b} F_{|a|c)} \\ D_{ab,yy} &= \frac{1}{3}\partial_a \partial_b A, \end{aligned} \tag{7.19}$$

while $D_{ay,yy}$ vanishes. Here we have denoted with $F_{ab} = \partial_{[a} A_{b]}$ the field-strength of the vector. As eq. (7.19) shows, the Maurer-Cartan form in three dimensions thus contains the Riemann tensor of A_{ab} together with the derivative of the field-strength of the vector and the double derivative of the scalar. This implies that among the rest, the Maurer-Cartan form is invariant under the transformations

$$\delta A_a = x^b b_{[ba]} \qquad \delta A = b_a x^a, \tag{7.20}$$

which indeed lead to

$$\delta F_{ab} = b_{[ab]} \qquad \delta(\partial_a A) = b_a. \tag{7.21}$$

Such transformations cannot be written as standard gauge transformations for the corresponding fields, and indeed they do not leave the field strength invariant, although they are symmetries of the dimensionally reduced Riemann tensor. They are generated by the operators associated to the $\overline{\text{Og}}$ 1 fields in figure 4, and in general we define the $\overline{\text{Og}}$ generators as those producing transformations that cannot be written as gauge transformations. The $\overline{\text{Og}}$ 1 fields in figure 4, together with the standard Og 1 fields for A_{ab} and A_a, are such

Figure 5. The A_1^{+++} Dynkin diagram.

that all the terms with one derivative of the fields in the Maurer-Cartan form vanish once the inverse Higgs mechanism is applied. The standard gauge transformations of the fields are obtained by performing a truncation that projects out the $\overline{\text{Og}}$ 1 generators, and once this truncation is performed one can no longer use the inverse Higgs mechanism to cancel the one derivative terms completely, which indeed give F_{ab} and the derivative of the scalar.

7.2 The dual graviton in A_1^{+++} in four dimensions

The non-linear realisation based on the algebra A_1^{+++}, whose Dynkin diagram is shown in figure 5, has the particular feature of only containing in four dimensions the graviton and its duals, which are fields with two symmetric indices together with an arbitrary number of blocks of two antisymmetric indices, as well as generators with sets of 3 or 4 antisymmetric indices. This in particular means that the spectrum does not contain any forms, that is fields with completely antisymmetric indices.

Decomposing the adjoint representation of A_1^{+++} in representations of $GL(4,\mathbb{R})$ one gets $K^a{}_b$ at level zero, which are the generators of $GL(4,\mathbb{R})$, and R^{ab} at level one. The generators at higher level can be obtained as multiple commutators of R^{ab} subject to the Serre relations, and the number of indices of a generator at level l is $2l$. We will ignore all the generators of level higher than 1 in this subsection. The commutation relation between $K^a{}_b$ and R^{ab} is

$$[K^a{}_b, R^{cd}] = \delta^c_b R^{ad} + \delta^d_b R^{ca}. \tag{7.22}$$

We now want to introduce the Og generators for both the graviton and the dual graviton, in order to reproduce the correct general coordinate transformation for the fields, as well as the expected gauge transformation for the dual graviton. As in the previous subsection, we have

$$[R^{ab}, P_c] = 0. \tag{7.23}$$

We thus obtain the commutator between the gravity Og 1 operator $K^{ab}{}_c$ and R^{ab} by the Jacobi identity with P_a. The result is

$$[K^{ab}{}_c, R^{de}] = \delta^d_c K_1^{e,ab} + \delta^e_c K_1^{d,ab} - 2(\delta^d_c \tilde{K}_1^{abe} + \delta^e_c \tilde{K}_1^{abd}). \tag{7.24}$$

Similarly, imposing the Jacobi identity between P_a, $K^a{}_b$ and the Og 1 dual graviton operators gives

$$\begin{aligned} [K^a{}_b, K_1^{c,de}] &= \delta^c_b K_1^{a,de} + \delta^d_b K_1^{c,ae} + \delta^e_b K_1^{c,da} \\ [K^a{}_b, \tilde{K}_1^{cde}] &= 3\delta_b^{(c} \tilde{K}_1^{de)a} \end{aligned} \tag{7.25}$$

as expected from the index structure of the operators.

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We would expect that the commutator between the Og 1 gravity operator $K^{ab}{}_c$ and the dual graviton Og 1 operators gave the Og 2 dual graviton operators in figure 3 if a description of both gravity and dual gravity were possible. This turns out to be impossible, i.e. one can show that the Jacobi identity between $K^{ab}{}_c$, P_a and either $K_1^{a,bc}$ or $\tilde{K}_1^{abc}$ is not satisfied if the commutator between $K^{ab}{}_c$ and the dual graviton Og 1 operators gives dual graviton Og 2 operators. This is indeed the problem that one typically encounters when trying to construct a dual Riemann tensor.

One can define an operator $\bar{K}_2^{ab,cd}$ satisfying

$$\bar{K}_2^{ab,cd} = \bar{K}_2^{(ab),cd} = \bar{K}_2^{ab,(cd)} = \bar{K}_2^{cd,ab} \qquad \bar{K}_2^{a(b,cd)} = 0, \tag{7.26}$$

whose corresponding Young tableaux is shown in the last column in figure 3. We define the commutation relation of this operator with P_a to be

$$[\bar{K}_2^{ab,cd}, P_e] = \frac{1}{2}\delta_e^{(a} K_1^{b),cd} + \frac{1}{2}\delta_e^{(c} K_1^{d),ab}. \tag{7.27}$$

This is indeed the most general result compatible with the symmetries, and one can show that the Jacobi identity with a further P_a operator is satisfied.

If one adds the term $\exp(\bar{\Phi}^2_{ab,cd}\bar{K}_2^{ab,cd})$ to the group element of eq. (7.12), one obtains that a transformation

$$\delta\bar{\Phi}^2_{ab,cd} = \bar{c}_{ab,cd} \tag{7.28}$$

implies an x^2 transformation for A_{ab} of the form

$$\delta A_{ab} = \frac{1}{2}\bar{c}_{ab,cd}x^c x^d. \tag{7.29}$$

This transformation cannot be written as a gauge transformation of eq. (7.8) for the linearised graviton. Following the arguments of the previous subsection, we refer to $\bar{K}_2^{ab,cd}$ as an $\overline{\text{Og}}$ operator. The inverse Higgs mechanism allows to gauge away completely all the terms at most quadratic in x in the field, with this still being compatible with all the symmetries.

Having introduced the operator $\bar{K}_2^{ab,cd}$, one obtains that the commutator between $K^{ab}{}_c$ and the dual graviton Og 1 operators can now be made compatible with the Jacobi identity with P_a. The result is

$$\begin{aligned}
[K^{ab}{}_c, K_1^{d,ef}] &= -6\delta_c^{(e} K_2^{|d|,f)ab} + 6\delta_c^{(d} K_2^{e,f)ab} + 2\delta_c^d \tilde{K}_2^{efab} - 2\delta_c^{(d}\tilde{K}_2^{ef)ab} \\
&\quad -4\delta_c^d \bar{K}_2^{ef,ab} + 4\delta_c^{(d}\bar{K}_2^{ef),ab} \\
[K^{ab}{}_c, \tilde{K}_1^{def}] &= 3\delta_c^{(d} K_2^{e,f)ab} - 10\delta_c^{(d}\tilde{K}_2^{ef)ab} + 2\delta_c^{(d}\bar{K}_2^{ef),ab}.
\end{aligned} \tag{7.30}$$

The fact that $\bar{K}_2^{ab,cd}$ must appear on the right hand side of this commutation relation is the main result of this section. This shows that the only inverse Higgs mechanism compatible with the symmetries is the one that gauges away the dual graviton completely. We expect that once all the $\overline{\text{Og}}$ operators for the dual graviton are introduced together with the Og operators for both the graviton and the dual graviton, the resulting algebra is well defined. We conjecture that the same applies to all the generators of A_1^{+++} with positive level. As a consequence of this, after having applied the inverse Higgs mechanism, the Maurer-Cartan form will contain only the graviton Riemann tensor and its derivatives.

It is interesting to discuss the dimensional reduction to three dimensions in this case as we have done in the previous subsection. Following arguments similar to that case, one can show that the dimensional reduction of all the generators in figure 3 contains the $\overline{\text{Og}}$ 1 and $\overline{\text{Og}}$ 2 generators for the scalar and the vector, and more generally the dimensional reduction of all the Og and $\overline{\text{Og}}$ dual graviton generators leads to all the Og and $\overline{\text{Og}}$ generators for the dimensionally reduced fields. The algebra of the dimensionally reduced theory can be truncated in such a way that the $\overline{\text{Og}}$ generators for the scalar and the vector that arise from the reduction of the dual graviton can be consistently projected out, so that the corresponding Maurer-Cartan form results in the field-strengths for this fields, as well as their derivatives, once the inverse Higgs mechanism is applied.

7.3 The dual graviton in E_8^{+++} in four dimensions

In this subsection we want to discuss the case in which the dual graviton couples to matter. We will discuss the case of the non-linear realisation of E_8^{+++}, i.e. E_{11}, in four dimensions, which corresponds to the bosonic sector of four-dimensional maximal supergravity. The Dynkin diagram of E_{11} is shown in figure 1, and the four dimensional theory is obtained deleting node 4 in the diagram. The internal symmetry is E_7, and the spectrum contains among the rest vectors in the **56** of E_7. We will show that even neglecting couplings to gravity, it is impossible to make the gauge transformation of the dual graviton compatible with that of the vector. The situation is exactly as in the previous subsection: the commutator of two Og 1 operators for the vector generate the operator $\bar{K}_2^{ab,cd}$, which is an $\overline{\text{Og}}$ 2 operator for the dual graviton.

Decomposing the adjoint representation of E_{11} in representations of $GL(4,\mathbb{R})$ one gets at level zero the gravity generators $K^a{}_b$ and the E_7 generators R^α, while at level 1 one gets $R^{a,M}$, where M denotes the **56** of E_7. The higher level generators can be obtained as multiple commutators of $R^{a,M}$. In particular at level 2 one gets

$$[R^{a,M}, R^{b,N}] = D_\alpha^{MN} R^{[ab],\alpha} + \Omega^{MN} R^{ab}, \tag{7.31}$$

where $R^{ab,\alpha}$ is the 2-form generator in the adjoint of E_7 and R^{ab} is the dual graviton generator. We have also introduced

$$D^{\alpha MN} = \Omega^{MP} D_P^{\alpha\, N} \tag{7.32}$$

which is symmetric in MN, and $D_M^{\alpha\, N}$ are the generators in the **56**. Finally Ω^{MN} is the antisymmetric invariant tensor of E_7. The field associated to the generator $R^{[ab],\alpha}$ is related to the scalars by duality. In the rest of this section we will ignore the 2-form contribution to the commutator of eq. (7.31), and we will only consider the dual graviton contribution,

$$[R^{a,M}, R^{b,N}] = \Omega^{MN} R^{ab}. \tag{7.33}$$

This truncation of the algebra is consistent because the Jacobi identities close independently on the 2-form generators and on the dual graviton generators.

The Og 1 generator for the vector $R^{a,M}$ is a generator $K^{ab,M}$ symmetric in ab, whose commutation relation with P_a is

$$[K^{ab,M}, P_c] = \delta_c^{(a} R^{b),M}. \tag{7.34}$$

The commutation relation of $K^{ab,M}$ with $R^{a,M}$ can be obtained by imposing the Jacobi identity of these operators with P_a and using eqs. (7.2), eq. (7.33) and eq. (7.34), as well as the fact that $R^{a,M}$ commutes with P_a. The result is

$$[R^{a,M}, K^{bc,N}] = -\frac{1}{2}\Omega^{MN} K_1^{a,bc} + \Omega^{MN} \tilde{K}_1^{abc}. \tag{7.35}$$

We can now write the group element up to Og 2 generators,

$$g = e^{x\cdot P} e^{\Phi^1_{a,bc} K_1^{a,bc}} e^{\tilde{\Phi}^1_{abc} \tilde{K}_1^{abc}} e^{\Phi_{ab,M} K^{ab,M}} e^{A_{ab} R^{ab}} e^{A_{a,M} R^{a,M}}, \tag{7.36}$$

which leads to the Maurer-Cartan form

$$\begin{aligned} g^{-1}\partial_\mu g = P_\mu + (\partial_\mu A_{a,M} - \Phi_{\mu a,M}) R^{a,M} + \Big(\partial_\mu A_{ab} + \frac{1}{2}\Omega^{MN} \partial_\mu A_{a,M} A_{b,N} \\ -\Phi^1_{\mu,ab} - \tilde{\Phi}^1_{\mu ab} - \Phi_{\mu a,M} A_{b,N} \Omega^{MN}\Big) R^{ab} + \cdots . \end{aligned} \tag{7.37}$$

The inverse Higgs mechanism then leaves the field strength for the vector, while the term contracting R^{ab} is put to zero by solving for $\Phi^1_{\mu,ab}$ and $\tilde{\Phi}^1_{\mu ab}$ in terms of A_{ab} and $A_{a,M}$.

We now compute the commutator of two Og 1 operators $K^{ab,M}$ for the vector, and we determine which Og 2 generators are needed to satisfy the Jacobi identities. It turns out that because of the symmetry of the commutator, it is not possible to generate the Og 2 dual graviton operator $K_2^{a,bcd}$, and the Jacobi identity with P_a imposes that this actually closes on $\tilde{K}_2^{abcd}$ and $\bar{K}_2^{ab,cd}$. The result is

$$[K^{ab,M}, K^{cd,N}] = 2\Omega^{MN} \tilde{K}_2^{abcd} - \Omega^{MN} \bar{K}_2^{ab,cd}. \tag{7.38}$$

Thus exactly as in the case of the dual graviton coupled to gravity of the previous subsection we have found here that the commutator of two Og operators generates an $\overline{\text{Og}}$ operator for the dual graviton, which means that a gauge invariant field strength for the dual graviton is not compatible with vector gauge invariance.

We claim that this is a generic feature of E_{11} positive level generators with spacetime indices with mixed symmetry. The algebra of their Og generators does not close, and one is forced to introduce $\overline{\text{Og}}$ generators for all these mixed symmetry generators. Only for the gravity generator, which has level zero, and for the generators with completely antisymmetric indices the Og algebra closes. As a consequence only for these fields can one use the inverse Higgs mechanism and be left with a non-vanishing field-strength. The fact that the positive level mixed symmetry generators require the introduction of the Og and $\overline{\text{Og}}$ generators implies instead that the corresponding fields do not allow a gauge invariant field strength and the inverse Higgs mechanism gauges away these fields completely. To show this one computes Jacobi identities involving positive level E_{11} generators, Og generators and the momentum operator P_a. Thus this result deeply relies on the structure of the E_{11} algebra. The dimensional reduction allows a further truncation of the algebra in the case in which a mixed symmetry generator gives rise to a generator with completely antisymmetric indices. Indeed in this case, as was shown in the previous subsections, the $\overline{\mathrm{Og}}$ generators can be consistently projected out.

It is important to stress that the dynamics is compatible with this construction. The field strengths of the antisymmetric fields are first order in derivatives, and therefore one needs fields and dual fields to construct duality relations which are first order equations for these fields. The gravity Riemann tensor instead is at second order in derivatives and thus there is no need of a dual field to construct its equation of motion.

8 Conclusions

In this paper we have given a method of obtaining field strengths and gauge transformations of all the massless and massive maximal supergravity theories starting from E_{11}. The global E_{11} transformations of the fields are promoted to gauge transformations by the inclusion in the algebra of additional generators.

We have first shown how this mechanism works for pure gravity. We have constructed Einstein's theory of gravity using a non-linear realisation which takes as its underlying algebra one that consists of $IGL(D,\mathbb{R})$ and an infinite set of additional generators whose effect is to promote the rigid $IGL(D,\mathbb{R})$ to be local. This infinite number of additional generators lead to local translations, that is general coordinate transformations, but to no new fields in the final theory as their Goldstone fields are solved in terms of the graviton field using a set of invariant constraints placed on the Cartan forms. This is an example of what has been called the inverse Higgs effect [23].

We have then generalised this procedure to E_{11} at low levels. We have taken the algebra, called E_{11}^{local} consisting of non-negative level E_{11} generators, the generators P_a and an infinite number of additional generators, whose role is to promote all the low level E_{11} symmetries to gauge symmetries. Again, as in the gravity case these generators do not lead to new Goldstone fields. We have shown that the non-linear realisation of the algebra E_{11}^{local} describes at low levels in eleven dimensions the 3-form and the 6-form of the eleven dimensional supergravity theory with all their gauge symmetries.

We have then considered in general the formulation of D-dimensional maximal gauged supergravity theories from the viewpoint of the enlarged algebra $\tilde{E}_{11,D}^{\mathrm{local}}$. We have first considered as a toy model the Scherk-Schwarz dimensional reduction of the IIB supergravity theory from this viewpoint. One starts from the algebra $E_{11,10B}^{\mathrm{local}}$ corresponding to the IIB theory and take the ten dimensional space-time to arise from an operator $\tilde{Q}$ which is constructed from $Q = P_9$ and part of the $\mathrm{SL}(2,\mathbb{R})$ symmetry of the theory. This means that the 10th direction of space-time is twisted to contain a part in the $\mathrm{SL}(2,\mathbb{R})$ coset symmetry of the theory. This non-linear realisation gives a nine dimensional gauged supergravity. We have observed that not all of the algebra $E_{11,10B}^{\mathrm{local}}$ is essential for the construction of the gauged supergravity in nine dimensions, but only an algebra which we call $\tilde{E}_{11,9}^{\mathrm{local}}$ which is the subalgebra of $E_{11,10B}^{\mathrm{local}}$ that commutes with $\tilde{Q}$. Its generators are non-trivial combinations of E_{11} generators and the additional generators and in general the generators of $\tilde{E}_{11,9}^{\rm local}$ have non-trivial commutation relations with nine dimensional space-time translations. Although the subalgebra $\tilde{E}_{11,9}^{\rm local}$ appears to be a deformation of the original E_{11} algebra and the space-time translations we have not changed the original commutators, but rather the new algebra arises due to the presence of the additional generators which are added to the E_{11} generators.

However, we have then shown that one can find the algebra $\tilde{E}_{11,9}^{\rm local}$ without carrying out all the above steps. Given the non-trivial relation between the lowest non-trivial positive level generator of $\tilde{E}_{11,9}^{\rm local}$ and the nine dimensional space-time translations one can derive the rest of the algebra $\tilde{E}_{11,9}^{\rm local}$ simply using Jacobi identities. This algebra determines uniquely all the field strengths of the theory, and thus one finds a very quick way of deriving the gauged supergravity theory.

This picture applies to all gauged supergravity theories, as one can easily find the algebra $\tilde{E}_{11,D}^{\rm local}$ without using its derivation from $E_{11}^{\rm local}$ and this provides a very efficient method of constructing all gauged supergravities. We have illustrated how this works by constructing the massive IIA theory as well as all the gauged maximal supergravities in five dimensions.

Finally, we have considered how this construction generalises to the fields with mixed symmetry, i.e. not completely antisymmetric, of E_{11} and in general of any non-linear realisation of a very-extended Kac-Moody algebra. We have considered as a prototype of such fields the dual graviton in four dimensions. If one tries to promote the global shift symmetry of the dual graviton field to a gauge symmetry, one finds that this is not compatible with the E_{11} algebra. The solution of this problem is that actually E_{11} forces to include additional generators, whose role is to enlarge the gauge symmetry of the dual graviton so that one can gauge away the field completely. This also applies if one only restricts his attention to the compatibility of the dual graviton with matter fields, that is if one neglects the gravity generators. This result agrees with the field theory analysis of [37]. More generally, this agrees with the no-go theorems of [38] on the consistency of self-interactions for the dual graviton. Recently, an alternative approach to the construction of an action for the dual graviton has been taken [39], in which the metric only appears via topological couplings, and an additional shift gauge field is included.

As we have mentioned in the introduction it is not obvious how to to implement the conformal group, or equivalently, add the Og fields in the presence of the generators of the l multiplet. The rational for introducing the l multiplet was that it would allow an E_{11} way of encoding space-time. However, in this paper we have chosen to take only the first component of the l multiplet, namely the space-time translations and we have taken this to commute with the positive level E_{11} generators. As a result we have had to discard the negative level E_{11} generators. This is unsatisfactory as E_{11} is defined from its Chevalley generators and relations and there is no definition that uses only the positive levels. For this reason the content of the adjoint representation and the l multiplet also rely on the negative root generators. However, we know that many of the generators, and so fields in the non-linear realisation, in the former and brane charges in the latter are in very convincing agreement with what one might expect in M theory. One example being the classification of all gauged supergravities using the $D-1$ forms found in the adjoint representation of E_{11}. How to reconcile local symmetries, space-time and the full E_{11} algebra is for future work.

Acknowledgments

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